

# A channel under simultaneous jamming and eavesdropping attack—correlated random coding capacities under strong secrecy criteria

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## Abstract

We give a complete characterization of the correlated random coding secrecy capacity of arbitrarily varying wiretap channels (AVWCs). We apply two alternative strong secrecy criteria, which both lead to the same multi-letter formula. The difference of these criteria lies in the treatment of correlated randomness, they coincide in the case of uncorrelated codes. On the basis of the derived formula, we show that the correlated random coding secrecy capacity is continuous as a function of the AVWC, in contrast to the discontinuous uncorrelated coding secrecy capacity. In the proof of the secrecy capacity formula for correlated random codes, we apply an auxiliary channel which is compound from the sender to the intended receiver and arbitrarily varying from the sender to the eavesdropper.

## I. INTRODUCTION

This paper brings together two areas of information theory: the arbitrarily varying channel (AVC) and the wiretap channel. This leads to the arbitrarily varying wiretap channel (AVWC): A sender would like to send information to a receiver through a noisy channel. Communication over this channel is subject to two difficulties. First, there is a second receiver, called an eavesdropper, which obtains its own noisy version of the channel inputs and should not be able to decode any information. Second, the state of the channels both to the intended receiver as well as to the eavesdropper can vary arbitrarily over time. Neither the sender nor the intended receiver know the true channel

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state. For a blocklength  $n$ , this means that the probability of the intended receiver obtaining the output sequence  $y^n = (y_1, \dots, y_n)$  and the eavesdropper receiving  $z^n = (z_1, \dots, z_n)$  given that  $x^n = (x_1, \dots, x_n)$  was input to the channel is contained in the family

$$\left\{ U_{s^n}^n(y^n, z^n | x^n) = \prod_{i=1}^n U_{s_i}(y_i, z_i | x_i) : s^n = (s_1, \dots, s_n) \in \mathcal{S}^n \right\}. \quad (1)$$

Here,  $\mathcal{S}$  is the finite state set and  $\{U_s(\cdot, \cdot | \cdot) : s \in \mathcal{S}\}$  a family of stochastic matrices, which thus determines the AVWC.

One could regard the varying channel states as determined by nature. However, we will interpret them as the result of jamming from an intruder. So henceforth, we shall view the AVWC as a channel under two attacks at the same time: one passive (eavesdropping), one active (jamming).

The study of correlated random coding capacities in their own right instead of as mathematical tools applied in the proofs of uncorrelated coding capacity theorems is motivated by arbitrarily varying channels (AVCs), which are AVWCs without the eavesdropper. By uncorrelated codes, we mean that sender and receiver have agreed on a procedure  $(f, \phi)$  of data manipulation prior to transmission. Here,  $f$  is a possibly stochastic mapping from the messages to the channel inputs of a fixed blocklength,  $\phi$  reverts channel outputs into messages. For transmission, each node separately executes its part of this procedure without relying on any further resources, in particular no common resources. What we call correlated random coding is usually called random coding and has been used as a mathematical tool ever since Shannon's 1948 paper [22]. Operationally, it means that sender and receiver agree on a family of deterministic codes  $\{(f^\gamma, \phi^\gamma) : \gamma \in \Gamma\}$ . Before communication, a random experiment following the distribution  $\mu$  on  $\Gamma$  is performed. The outcome, say  $\gamma$ , is revealed to sender and intended receiver which then apply the deterministic code  $(f^\gamma, \phi^\gamma)$ .

It was already observed by Blackwell, Breiman and Thomasian [7] that whether correlated randomness is available to sender and receiver can be crucial when it comes to the AVC capacity. In fact, AVCs exhibit a dichotomy [1]: Their capacity for deterministic coding either equals their capacity for correlated random coding or it equals zero. Csiszár and Narayan have identified the distinguishing property [12], called symmetrizability (a concept originally introduced by Ericson [15]). Without the use of correlated random coding, a symmetrizable AVC is useless; no message transmission is possible.

Thus one is led to regarding correlated randomness as an additional resource for communication. This resource can make communication possible where it is impossible without. Of course, it is important that the jammer has no access to this resource, i. e. that it does not know the outcome of the random experiment common to sender and receiver. In this paper, we will apply two strong secrecy criteria and show that the corresponding capacities for correlated random coding coincide. The first of these criteria is that

$$\max_{s^n} \sum_{\gamma} I(M \wedge Z_{s^n}^\gamma) \mu(\gamma) \quad (2)$$

be small, where  $M$  is the message chosen uniformly at random and  $Z_{s^n}^\gamma$  is the eavesdropper's output if the state sequence is  $s^n$  and the deterministic code  $(f^\gamma, \phi^\gamma)$  has been selected. This criterion was applied in [4], [20]. The

second, stronger one requires

$$\max_{s^n} \max_{\gamma} I(M \wedge Z_{s^n}^{\gamma}) \quad (3)$$

to be small. Both secrecy criteria assume that the eavesdropper knows the realization of the correlated randomness. This means that we have to assume the active and passive attacks to be uncoordinated in the sense that the eavesdropper does not inform the jammer about its knowledge of the correlated randomness.

We are not the first to study the capacity of the AVWC. A study of the Gaussian MIMO wiretap channel where the channel to the eavesdropper is arbitrarily varying has been done in [18], [19]. Earlier approaches to the discrete AVWC as defined in (1) can be found in [4], [20], which studied the secrecy capacity achieved by correlated random coding and used (2) as secrecy criterion. In both papers, closed-form secrecy capacity results could only be given after imposing additional conditions.

The main result of this paper will be a complete characterization of the correlated random coding secrecy capacity under both criteria (2) and (3). The capacity formula we find is multi-letter. It was found in [4] for special AVWCs where there is a “best channel to the eavesdropper” and reduces to a single-letter formula under certain degradedness conditions as required in [20]. It is not clear whether a generally applicable single-letter formula exists at all. Still, the multi-letter formula allows for the approximate computation of the secrecy capacity up to a given complexity. However, this is not our main concern, so we do not provide any relation between complexity and approximation goodness.

With the help of the multi-letter formula, it can also be shown that the correlated random coding secrecy capacity is continuous in the channel. Thus small errors in the description of the family (1) do not have severe consequences on the capacity. If the capacity formula were not continuous, the channel would in general have to be estimated with infinite precision in order to meaningfully apply the capacity formula. The continuity of the correlated random coding secrecy capacity becomes even more remarkable as very simple examples with  $|\mathcal{S}| = 2$  have been given in [9] which show that the uncorrelated coding secrecy capacity is a discontinuous function of the AVWC.

For the achievability part of the capacity theorem, we follow Ahlswede’s strategy of deriving correlated random coding achievability results for AVCs from uncorrelated coding capacity results for compound channels. (In contrast to an AVC, a compound channel does not change its state during the transmission of a codeword.) This technique is known as the “robustification technique”. Sender and receiver of an AVC randomly permute an uncorrelated code for a certain compound channel induced by the AVC and thus obtain a correlated random code with negligibly larger average error.

When applying the robustification technique to AVWCs, one has to take the secrecy criterion into account. As seen in [4], this requires a “best channel to the eavesdropper” if one assumes the channel to the eavesdropper to be compound as well. The central idea of our proof is to introduce the compound-arbitrarily varying wiretap channel (CAVWC). This channel is compound from sender to intended receiver and arbitrarily varying from sender to eavesdropper. We derive the uncorrelated coding secrecy capacity of this channel. After robustification, this also turns out to be the correlated random coding secrecy capacity of the AVWC.

We prove the achievability result for the CAVWC by random coding following Devetak [13]. This technique takes a resolvability approach to proving secrecy, cf. the discussion of resolvability and “capacity-based” approaches by Bloch and Laneman [8]. However, it does not follow an information spectrum approach like the techniques presented in [8]. To our knowledge, those techniques have not yet been shown to be able to handle arbitrarily varying channels. As the number of AVWC channel states grows exponentially with blocklength, very tight probability estimates have to be obtained from random coding. Devetak’s method [13], originally in the language of quantum information theory, provides such estimates and was already applied in [23] in a classical information theory setting.

In [10], an a priori upper bound on the amount of correlated randomness required to achieve the correlated random coding secrecy capacity was found. Such a bound is necessary for the converse of the correlated random coding secrecy capacity theorem for the AVWC. The reason for this is that the use of correlated randomness prohibits a straightforward application of the data processing inequality.

In a follow-up work [21] to this paper, the AVWC correlated random coding secrecy capacity for the case that the eavesdropper has no knowledge of the correlated randomness as well as the AVWC uncorrelated coding secrecy capacity are studied.

*Paper outline:* In Section II, we set the notation and give basic definitions. In Section III we define the AVWC and state the coding problem and the main result. Section IV discusses the main result of Section III. Section V introduces the CAVWC mentioned in the introduction, states the CAVWC coding problem and the corresponding secrecy capacity theorem. Section VI contains the proof of the achievability part of the coding theorem for the CAVWC. The achievability part of the correlated random coding theorem for the AVWC is derived from the achievability part of the coding theorem for the CAVWC in Section VII. Section VIII contains the converses. In Section IX, a short discussion concludes the paper. Several proofs are collected in the appendices.

## II. NOTATION AND BASIC DEFINITIONS

Logarithms denoted by  $\log$  are taken to the base 2; correspondingly, we set  $\exp(x) = 2^x$ . The cardinality of a finite set  $\mathcal{A}$  is written  $|\mathcal{A}|$ . For a subset  $\mathcal{E}$  of  $\mathcal{A}$ , we write  $\mathcal{E}^c := \mathcal{A} \setminus \mathcal{E}$ . The *indicator function*  $\mathbb{1}_{\mathcal{E}}$  assumes the value 1 for arguments contained in  $\mathcal{E}$  and 0 else. For  $n$ -tuples contained in  $\mathcal{A}^n$ , we write  $x^n := (x_1, \dots, x_n) \in \mathcal{A}^n$ .

The set of probability measures on the finite set  $\mathcal{A}$  is denoted by  $\mathcal{P}(\mathcal{A})$ . For  $P \in \mathcal{P}(\mathcal{A})$ , we define the  $n$ -fold product measure  $P^n \in \mathcal{P}(\mathcal{A}^n)$  by  $P^n(x^n) := \prod_i P(x_i)$ . We write stochastic matrices  $\{W(b|a) : a \in \mathcal{A}, b \in \mathcal{B}\}$  with input alphabet  $\mathcal{A}$  and output alphabet  $\mathcal{B}$  as mappings  $W : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{B})$ . A nonnegative measure on  $\mathcal{A}$  is a vector  $(\mu(a))_{a \in \mathcal{A}}$  with  $\mu(a) \geq 0$  for all  $a \in \mathcal{A}$ . A probability measure is a nonnegative measure. The total variation distance of two nonnegative measures  $\mu, \nu$  on  $\mathcal{A}$  is defined by  $\|\mu - \nu\| := \sum_{a \in \mathcal{A}} |\mu(a) - \nu(a)|$ .

If  $\bar{X}, \bar{Y}$  are random variables, then we write the distribution of  $\bar{X}$  as  $P_{\bar{X}}$ , the joint distribution of  $\bar{X}$  and  $\bar{Y}$  as  $P_{\bar{X}\bar{Y}}$  and the conditional distribution of  $\bar{X}$  given  $\bar{Y}$  as  $P_{\bar{X}|\bar{Y}}$ .

For a sequence  $x^n = (x_1, \dots, x_n) \in \mathcal{A}^n$  and  $a \in \mathcal{A}$ , the number  $N(a|x^n)$  indicates the number of coordinates  $x_i$  of  $x^n$  with  $x_i = a$ . The type of  $x^n$  is the probability measure  $q \in \mathcal{P}(\mathcal{A})$  defined by  $q(a) := N(a|x^n)/n$ . The set of all possible types of sequences of length  $n$  is denoted by  $\mathcal{P}_0^n(\mathcal{A})$ . For  $\delta > 0$  and an  $\mathcal{A}$ -valued random variable

$\bar{X}$ , we define the typical set  $\mathcal{T}_{\bar{X},\delta}^n \subset \mathcal{A}^n$  as the set of those  $x^n \in \mathcal{A}^n$  satisfying the two conditions

$$\left| \frac{1}{n} N(a|x^n) - P_{\bar{X}}(a) \right| < \delta \quad \text{for every } a \in \mathcal{A},$$

$$N(a|x^n) = 0 \text{ if } P_{\bar{X}}(a) = 0.$$

For  $\delta > 0$ , an  $\mathcal{A} \times \mathcal{B}$ -valued random variable  $(\bar{X}, \bar{Y})$  with joint distribution  $P_{\bar{X}\bar{Y}}$  and an element  $x^n$  of  $\mathcal{A}^n$ , we define the conditionally typical set  $\mathcal{T}_{\bar{Y}|\bar{X},\delta}^n(x^n)$  as the set of those  $y^n \in \mathcal{B}^n$  satisfying the two conditions

$$\left| \frac{1}{n} N(a, b|x^n, y^n) - P_{\bar{Y}|\bar{X}}(b|a) \frac{1}{n} N(a|x^n) \right| < \delta \quad \text{for all } a \in \mathcal{A}, b \in \mathcal{B},$$

$$N(a, b|x^n, y^n) = 0 \text{ if } P_{\bar{Y}|\bar{X}}(b|a) = 0.$$

### III. ARBITRARILY VARYING WIRETAP CHANNELS

Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$  be finite sets. For every  $s \in \mathcal{S}$ , let a stochastic matrix  $W_s : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{B})$  and another stochastic matrix  $V_s : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{C})$  be given. For a number  $n$  and  $x^n \in \mathcal{A}^n, y^n \in \mathcal{B}^n, s^n \in \mathcal{S}^n$ , define

$$W_{s^n}^n(y^n|x^n) := \prod_{i=1}^n W_{s_i}(y_i|x_i).$$

We denote the family  $\{W_{s^n}^n : s^n \in \mathcal{S}^n, n = 1, 2, \dots\}$  by  $\mathfrak{W}$ . In analogy to  $W_{s^n}^n(y^n|x^n)$ , we define  $V_{s^n}^n(z^n|x^n)$  for  $z^n \in \mathcal{C}^n$  and denote the corresponding family  $\{V_{s^n}^n : s^n \in \mathcal{S}^n, n = 1, 2, \dots\}$  by  $\mathfrak{V}$ . We sometimes prefer to write  $V^n(z^n|x^n, s^n)$  instead of  $V_{s^n}^n(z^n|x^n)$ . We call the pair  $(\mathfrak{W}, \mathfrak{V})$  an *Arbitrarily Varying Wiretap Channel (AVWC)*.  $\mathcal{S}$  is called the *state set* of  $(\mathfrak{W}, \mathfrak{V})$ .

*Remark 1:* One checks easily that the representation of an AVWC as a pair  $(\mathfrak{W}, \mathfrak{V})$  is possible without losing generality. In general, any state  $s \in \mathcal{S}$  together with an input  $a \in \mathcal{A}$  will lead to a joint output distribution  $U_s(\cdot, \cdot|a)$ . But the performance of any of the codes defined below is measured with respect to the marginal output distributions  $W_s(\cdot|a)$  and  $V_s(\cdot|a)$ . Thus for the purpose of this paper, all AVWCs with the same marginals  $\mathfrak{W}$  and  $\mathfrak{V}$  are equivalent.

An uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n$  for the AVWC  $(\mathfrak{W}, \mathfrak{V})$  consists of a stochastic encoder  $E : \{1, \dots, J_n\} \rightarrow \mathcal{P}(\mathcal{A}^n)$  and a collection of mutually disjoint sets  $\{\mathcal{D}_j \subset \mathcal{B}^n : 1 \leq j \leq J_n\}$  whose union equals  $\mathcal{B}^n$ . We abbreviate  $\mathcal{J}_n := \{1, \dots, J_n\}$ . Together with an AVWC  $(\mathfrak{W}, \mathfrak{V})$ , any uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n$  defines a canonical family

$$\mathcal{F}(\mathcal{K}_n, \mathfrak{W}, \mathfrak{V}) := \{M^n, X^n, Y_{s^n}^n, Z_{s^n}^n, \hat{M}_{s^n}^n : s^n \in \mathcal{S}^n\} \quad (4)$$

of random variables, with  $M^n$  and  $\hat{M}_{s^n}^n$  assuming values in  $\mathcal{J}_n$ , the values of  $X^n$  in  $\mathcal{A}^n$ , those of  $Y_{s^n}^n$  in  $\mathcal{B}^n$ , those of  $Z_{s^n}^n$  in  $\mathcal{C}^n$ , and such that for every  $s^n \in \mathcal{S}^n$  the distribution of  $(M^n, X^n, Y_{s^n}^n, Z_{s^n}^n, \hat{M}_{s^n}^n)$  equals

$$P_{M^n X^n Y_{s^n}^n Z_{s^n}^n \hat{M}_{s^n}^n}(j, x^n, y^n, z^n, \hat{j}) = \frac{1}{J_n} E(x^n|j) W_{s^n}^n(y^n|x^n) V_{s^n}^n(z^n|x^n) \mathbb{1}_{\mathcal{D}_j}(y^n).$$

Recall that we incur no loss of generality by defining  $Y_{s^n}^n$  and  $Z_{s^n}^n$  to be independent conditional on  $X^n$ , as the joint distribution of  $Y_{s^n}^n$  and  $Z_{s^n}^n$  will never play any role (cf. Remark 1). The average error of  $\mathcal{K}_n$  is given by

$$e(\mathcal{K}_n) := \max_{s^n \in \mathcal{S}^n} \mathbb{P}[M^n \neq \hat{M}_{s^n}^n].$$

*Definition 2:* A non-negative number  $R_S$  is an *achievable uncorrelated coding secrecy rate* for the AVWC  $(\mathfrak{W}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n)_{n=1}^\infty$  of uncorrelated  $(n, J_n)$ -codes such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log J_n \geq R_S, \quad (5)$$

$$\lim_{n \rightarrow \infty} e(\mathcal{K}_n) = 0, \quad (6)$$

$$\lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n) = 0. \quad (7)$$

The *uncorrelated coding secrecy capacity* of  $(\mathfrak{W}, \mathfrak{V})$  is the supremum of all achievable secrecy rates  $R_S$  and is denoted by  $C_S(\mathfrak{W}, \mathfrak{V})$ .

Note the different roles the families  $\mathfrak{W}$  and  $\mathfrak{V}$  play.  $\mathfrak{W}$  is an *Arbitrarily Varying Channel (AVC)* from a sender with alphabet  $\mathcal{A}$  to a receiver with alphabet  $\mathcal{B}$ . Messages are supposed to be sent over this AVC in such a way that only a small, asymptotically negligible average error is incurred. This is reflected in condition (6). This communication is subject to an additional secrecy condition. An eavesdropper obtains a noisy version of the sender's channel inputs via the AVC  $\mathfrak{V}$ . Condition (7) guarantees secrecy no matter what the channel state is.

For given  $(n, J_n)$ , we assume that the set of uncorrelated  $(n, J_n)$ -codes is indexed by the set  $\Gamma_n$ . That means that the set of all uncorrelated  $(n, J_n)$ -codes (with given channel input and output alphabets  $\mathcal{A}$  and  $\mathcal{B}$ ) has the form  $\{\mathcal{K}_n(\gamma) : \gamma \in \Gamma_n\}$ . For the uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n(\gamma)$ , with  $\gamma \in \Gamma_n$ , we write for the canonical family of random variables

$$\mathcal{F}(\mathcal{K}_n(\gamma), \mathfrak{W}, \mathfrak{V}) = \{M^n, X^n(\gamma), Y_{s^n}^n(\gamma), Z_{s^n}^n(\gamma), \hat{M}_{s^n}^n(\gamma) : s^n \in \mathcal{S}^n, \gamma \in \Gamma_n\}.$$

A correlated random  $(n, J_n)$ -code  $\mathcal{K}_n^{\text{ran}}$  for the AVWC  $(\mathfrak{W}, \mathfrak{V})$  then is given by a finitely supported<sup>1</sup> random variable  $G_n$  on  $\Gamma_n$  independent of all canonical families of random variables  $\mathcal{F}(\mathcal{K}_n(\gamma), \mathfrak{W}, \mathfrak{V})$ . In other words,  $G_n$  randomly chooses an uncorrelated  $(n, J_n)$ -code out of all possible ones and is independent of the message random variable, the randomness in the chosen stochastic encoder and the channel noise. The average error  $e(\mathcal{K}_n^{\text{ran}})$  is defined as

$$e(\mathcal{K}_n^{\text{ran}}) := \max_{s^n \in \mathcal{S}^n} \mathbb{P}[M^n \neq \hat{M}_{s^n}^n(G_n)] = \max_{s^n \in \mathcal{S}^n} \sum_{\gamma \in \Gamma_n} \mathbb{P}[M^n \neq \hat{M}_{s^n}^n(\gamma)] P_{G_n}(\gamma),$$

where  $\sum_{\gamma \in \Gamma_n} a(\gamma) P_{G_n}(\gamma)$  is short for the finite sum  $\sum_{\gamma \in \text{supp}(G_n)} a(\gamma) P_{G_n}(\gamma)$ .

In the case of correlated random codes, we consider two secrecy criteria, leading to two different notions of achievable rate.

*Definition 3:* A non-negative number  $R_S$  is called an *achievable correlated random coding mean secrecy rate*

<sup>1</sup>“Finitely supported” means that the set  $\text{supp}(G_n) := \{\gamma \in \Gamma_n : P_{G_n}(\gamma) > 0\}$  called the *support* of  $G_n$  is finite.

for the AVWC  $(\mathfrak{W}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n^{\text{ran}})_{n=1}^\infty$  of correlated random  $(n, J_n)$ -codes such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log J_n \geq R_S, \quad (8)$$

$$\lim_{n \rightarrow \infty} e(\mathcal{K}_n^{\text{ran}}) = 0, \quad (9)$$

$$\lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n | G_n) = 0. \quad (10)$$

The supremum of all achievable secrecy rates for correlated random codes is called the *correlated random coding mean secrecy capacity* of  $(\mathfrak{W}, \mathfrak{V})$  and denoted by  $C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$ .

*Definition 4:* A non-negative number  $R_S$  is called an *achievable correlated random coding maximal secrecy rate* for the AVWC  $(\mathfrak{W}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n^{\text{ran}})_{n=1}^\infty$  of correlated random  $(n, J_n)$ -codes such that (8) and (9) hold and

$$\lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} \max_{\gamma \in \text{supp}(G_n)} I(M^n \wedge Z_{s^n}^n(\gamma)) = 0. \quad (11)$$

The supremum of all achievable correlated random coding maximal secrecy rates is called the *correlated random coding maximal secrecy capacity* of  $(\mathfrak{W}, \mathfrak{V})$  and denoted by  $C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V})$ .

*Remark 5:* It is immediately clear that  $C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V}) \geq C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V})$ .

The secrecy capacities for correlated random codes are characterized by a multi-letter formula, extending the results of [4]. We set

$$R_S^*(\mathfrak{W}, \mathfrak{V}) := \lim_{k \rightarrow \infty} \frac{1}{k} \sup_{\{\bar{U}, \bar{X}^k, \bar{Y}_q^k, \bar{Z}_{s^k}^k\}} \left( \min_{q \in \mathcal{P}(\mathcal{S})} I(\bar{U} \wedge \bar{Y}_q^k) - \max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k) \right) \quad (12)$$

where the supremum is over the set of families of random variables

$$\{\bar{U}, \bar{X}^k, \bar{Y}_q^k, \bar{Z}_{s^k}^k : q \in \mathcal{P}(\mathcal{S}), s^k \in \mathcal{S}^k\} \quad (13)$$

satisfying that  $\bar{U}$  assumes values in some finite subset of the integers, the values of  $\bar{X}^k$  lie in  $\mathcal{A}^k$ , those of  $\bar{Y}_q^k$  in  $\mathcal{B}^k$ , those of  $\bar{Z}_{s^k}^k$  in  $\mathcal{C}^k$ , and such that for every  $q \in \mathcal{P}(\mathcal{S})$  and  $s^k \in \mathcal{S}^k$ ,

$$P_{\bar{U} \bar{X}^k \bar{Y}_q^k \bar{Z}_{s^k}^k}(u, x^k, y^k, z^k) = P_{\bar{U}}(u) P_{\bar{X}^k | \bar{U}}(x^k | u) \left( \prod_{i=1}^k \left[ \sum_{s \in \mathcal{S}} q(s) W_s(y_i | x_i) \right] \right) V_{s^k}^k(z^k | x^k). \quad (14)$$

$P_{\bar{U}}$  and  $P_{\bar{X}^k | \bar{U}}$  may be arbitrary probability distributions and stochastic matrices, respectively.

*Theorem 6:* For the AVWC  $(\mathfrak{W}, \mathfrak{V})$ , we have

$$C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V}) = C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V}) = R_S^*(\mathfrak{W}, \mathfrak{V}).$$

- Remark 7:*
- 1) It is shown exactly as in [5], using Fekete's lemma [16], that the limit on the right-hand side of (12) indeed exists. In fact, the limit can be replaced by a supremum, as the terms  $\frac{1}{k} \sup(\dots)$  increase in  $k$ .
  - 2) For given  $k$ , the cardinality of  $\mathcal{U}$  can be restricted to  $|\mathcal{A}|^k$ . This can be proved almost exactly as in the proof of [11, Theorem 17.11]. The supremum in (12) then becomes a maximum.
  - 3) If for  $q \in \mathcal{P}(\mathcal{S})$  we define  $W_q(b|a) := \sum_s q(s) W_s(b|a)$ , the conditional probability of  $\bar{Y}_q^k$  given  $\bar{X}^k$  in (14) satisfies

$$P_{\bar{Y}_q^k | \bar{X}^k}(y^k | x^k) = \prod_{k=1}^k W_q(y_i | x_i) =: W_q^k(y^k | x^k).$$



The family  $\{W_q^n : q \in \mathcal{P}(\mathcal{S}), n = 1, 2, \dots\}$  is a memoryless channel which does not change its state during the transmission of a codeword. Such channels will appear later under the name of *compound channel*.

4) The work [21] following up on this paper makes use of the fact that

$$R_S^*(\mathfrak{W}, \mathfrak{V}) = \lim_{k \rightarrow \infty} \frac{1}{k} \sup_{\{\bar{U}, \bar{X}^k, \bar{Y}_{\tilde{q}}, \bar{Z}_{s^k}^k\}} \left( \min_{\tilde{q} \in \mathcal{P}(\mathcal{S}^k)} I(\bar{U} \wedge \bar{Y}_{\tilde{q}}^k) - \max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k) \right) \quad (15)$$

$$= \lim_{k \rightarrow \infty} \frac{1}{k} \sup_{\{\bar{U}, \bar{X}^k, \bar{Y}_{\tilde{q}_1}, \bar{Z}_{\tilde{q}_2}^k\}} \left( \min_{\tilde{q}_1 \in \mathcal{P}(\mathcal{S}^k)} I(\bar{U} \wedge \bar{Y}_{\tilde{q}_1}^k) - \max_{\tilde{q}_2 \in \mathcal{P}(\mathcal{S}^k)} I(\bar{U} \wedge \bar{Z}_{\tilde{q}_2}^k) \right), \quad (16)$$

where the family of random variables in (16) is defined analogously to the family (13) with the difference that the parameters  $\tilde{q}_1, \tilde{q}_2$  range over all probability distributions on  $\mathcal{P}(\mathcal{S}^k)$  (in particular, not just the product measures with constant marginals or the extremal Dirac distributions) and where for  $\tilde{q}_1, \tilde{q}_2 \in \mathcal{P}(\mathcal{S}^k)$

$$P_{\bar{Y}_{\tilde{q}_1}^k \bar{Z}_{\tilde{q}_2}^k | \bar{X}}(y^k, z^k | x^k) = \left( \sum_{s^k} \tilde{q}_1(s^k) W_{s^k}^k(y^k | x^k) \right) \left( \sum_{s^k} \tilde{q}_2(s^k) V_{s^k}^k(z^k | x^k) \right).$$

The family of random variables in (15) over which the supremum is taken is obtained by restricting the parameters  $\tilde{q}_2$  in the family of random variables in (16) to the extremal Dirac measures, which means nothing else than to take  $P_{\bar{Z}_{s^k}^k | \bar{X}}$  as in (13). Similarly, by restricting the  $\tilde{q}_1$  to be product measures on  $\mathcal{S}^k$  with constant marginals, one can regard (13) itself as a restriction of the family in (15).

To prove the equalities (15) and (16), first note that due to the convexity of mutual information in the channel nothing changes if  $\max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k)$  on the right-hand side of (15) is replaced by  $\max_{\tilde{q}_2 \in \mathcal{P}(\mathcal{S}^k)} I(\bar{U} \wedge \bar{Z}_{\tilde{q}_2}^k)$ . This proves equality in (16). It is also obvious that the right-hand side of (15) is a lower bound on  $R_S^*(\mathfrak{W}, \mathfrak{V})$ . That equality holds can be seen by inspection of the proof of the converse in Section VIII below. The main reason is the fact that the average decoding error for AVC and AVWC is affine in the channel, as proved in [11, Lemma 12.3]. More details on this can be found in Remark 17 after the proof of the converse.

The enlargement of the state space as in (15) and (16) can be interpreted as allowing randomized jamming strategies. This does not affect the AVWC performance because the performance measures are robust against this randomization (i. e. the average error is affine in the channel, mutual information between the message and the eavesdropper's output is even convex in the channel).

5) Comparison of the right-hand side of (12) with the capacity expressions derived in [8] suggests that the terms  $\min_{q \in \mathcal{P}(\mathcal{S})} I(\bar{U} \wedge \bar{Y}_q^k)$  are related to an inf-information rate for the AVC  $\mathfrak{W}$  and  $\max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k)$  to a sup-information rate for the AVC  $\mathfrak{V}$ , see also [17]. However, as AVCs have not yet been treated in the framework of the theory of information spectrum, this remains speculation for the time being.

#### IV. DISCUSSION OF THEOREM 6

##### A. Multi-letter vs. single-letter

The bound from Remark 7-2) on the size of  $\mathcal{U}$  for fixed  $k$  does not give a general upper bound on the cardinality of the auxiliary alphabet  $\mathcal{U}$ . It could still be helpful in calculations of  $R_S^*(\mathfrak{W}, \mathfrak{V})$  if one knows from other arguments



that there exists a  $k_0$  such that, for  $k \geq k_0$ ,

$$\frac{1}{k} \sup_{\{\bar{U}, \bar{X}^k, \bar{Y}_q^k, \bar{Z}_{s^k}^k\}} \left( I(\bar{U} \wedge \bar{Y}_q^k) - \max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k) \right)$$

is sufficiently close to  $R_S^*(\mathfrak{W}, \mathfrak{V})$ . From Remark 7-1) it follows that this approach would give a lower bound on the secrecy capacity. Note that it is not at all clear whether a single-letter characterization of  $R_S^*(\mathfrak{W}, \mathfrak{V})$  is available. In the case of the unavailability of a single-letter capacity expression, only approximate calculations of capacity are possible.

That the above multi-letter characterization can lead to further insights into the nature of AVWCs can be seen in Subsection IV-C, where the continuity of  $R_S^*(\mathfrak{W}, \mathfrak{V})$  in  $(\mathfrak{W}, \mathfrak{V})$  is shown. To show this a priori, i. e. without having the multi-letter expression for capacity, seems to be very hard. With the formula at hand, however, it can be done. For the uncorrelated coding secrecy capacity, a similar study of continuity is performed in [21], also on the basis of the multi-letter formula.

A single-letter formula for  $C_{S, \text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$  has been given in [20] for AVWCs which satisfy certain conditions. We now present these conditions and show that if they are satisfied, the formula found in [20] coincides with  $R_S^*(\mathfrak{W}, \mathfrak{V})$ , which then becomes single-letter.

The first condition of [20] is that  $(\mathfrak{W}, \mathfrak{V})$  be *strongly degraded with independent states*. This means

- that  $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$  and that the families  $\{W_{(s_1, s_2)} : (s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2\}$  and  $\{V_{(s_1, s_2)} : (s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2\}$  of stochastic matrices determining  $\mathfrak{W}$  and  $\mathfrak{V}$  satisfy  $W_{(s_1, s_2)} = W_{s_1}$  and  $V_{(s_1, s_2)} = V_{s_2}$  for all  $(s_1, s_2)$ ; and
- that for every  $q_1 \in \mathcal{P}(\mathcal{S}_1)$  and  $q_2 \in \mathcal{P}(\mathcal{S}_2)$ , the matrix  $V_{q_2}$  should be a degraded version of  $W_{q_1}$ , where

$$W_{q_1}(y|x) = \sum_{s_1 \in \mathcal{S}_1} W_{s_1}(y|x)q_1(s_1), \quad V_{q_2}(z|x) = \sum_{s_2 \in \mathcal{S}_2} V_{s_2}(z|x)q_2(s_2),$$

and  $V_{q_2}$  is a degraded version of  $W_{q_1}$  if there exists a stochastic matrix  $T_{q_1 q_2} : \mathcal{B} \rightarrow \mathcal{C}$  such that

$$V_{q_2}(z|x) = \sum_y T_{q_1 q_2}(z|y) W_{q_1}(y|x). \quad (17)$$

(Observe: It is sufficient to require (17) to hold only for  $s_2 \in \mathcal{S}_2$  and  $q_1 \in \mathcal{P}(\mathcal{S}_1)$ . The validity of (17) for all  $q_1 \in \mathcal{P}(\mathcal{S}_1)$  and  $q_2 \in \mathcal{P}(\mathcal{S}_2)$  then follows upon setting  $T_{q_1 q_2}(z|y) := \sum_{s_2} q_2(s_2) T_{q_1 s_2}(z|y)$  for all  $y \in \mathcal{B}, z \in \mathcal{C}$ . Thus the function  $(q_1, q_2) \mapsto T_{q_1 q_2}$  can without loss of generality be assumed to be linear in  $q_2$ .

This is not possible for  $q_1$ , as can be seen from analyzing Example 3 in [20].)

The second condition of [20] is essentially the *best channel to the eavesdropper* condition from [4], so we will henceforth call it this way. It requires that there exists an  $s_* \in \mathcal{S}_2$  such that for all  $s_2 \in \mathcal{S}_2$ , the channel  $V_{s_2}$  is a degraded version of  $V_{s_*}$ , with degradedness here defined analogously to (17). (The general definition of “best channel to the eavesdropper” in [4], [20] does not require independent states.)

*Corollary 1:* If the AVWC  $(\mathfrak{W}, \mathfrak{V})$  is strongly degraded with independent states and has a best channel to the eavesdropper, then

$$R_S^*(\mathfrak{W}, \mathfrak{V}) = \max_{\{\bar{X}, \bar{Y}_{q_1}, \bar{Z}_{s_2}\}} \left( \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} I(\bar{X} \wedge \bar{Y}_{q_1}) - \max_{s_2 \in \mathcal{S}_2} I(\bar{X} \wedge \bar{Z}_{s_2}) \right) \quad (18)$$

where the maximum over  $\{\bar{X}, \bar{Y}_{q_1}, \bar{Z}_{s_2}\}$  is over families of random values satisfying

$$P_{\bar{X}\bar{Y}_{q_1}\bar{Z}_{s_2}}(x, y, z) = P_{\bar{X}}(x)W_{q_1}(y|x)V_{s_2}(z|x)$$

and where  $\bar{X}$  is an arbitrary  $\mathcal{A}$ -valued random variable.

*Proof:* See Appendix A. ■

### B. The amount of correlated randomness

Next we ask how many values the correlated randomness variable should attain with positive probability in order for  $C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$  and  $C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V})$  to be achievable. This can be answered in an a priori fashion, so it can be applied in the converse of Theorem 6.

Note that the definitions allow every kind of correlated randomness as long as it is finitely supported. In the achievability proof of Theorem 6, we shall see that the uniform distribution on a set of cardinality  $n!$  is sufficient, where  $n$  is the blocklength of the code. The size of this set can still be reduced considerably. For AVCs, the first such reduction was presented by Ahlswede in [1], where he showed that  $|\text{supp}(G_n)| \leq n^{1+\varepsilon}$  is sufficient.

A stronger result has been found recently [10]. Its essence is that every secrecy rate  $R_S < C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V})$  is achievable with no more than a finite amount of correlated randomness, given arbitrary upper bounds on the average error and the mutual information between message random variable and eavesdropper output.

*Lemma 8 ([10]):* Let  $R_S < C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$  and  $\lambda, \delta > 0$ . Then for every  $\varepsilon > 0$  there exists a positive integer  $L = L(R_S, \varepsilon, \lambda, \delta)$  such that for sufficiently large  $n$  there exists a correlated random  $(n, J_n)$ -code  $\mathcal{K}_n^{\text{ran}}$  satisfying

$$\frac{1}{n} \log J_n \geq R_S - \varepsilon, \quad (19)$$

$$e(\mathcal{K}_n^{\text{ran}}) \leq \lambda, \quad (20)$$

$$\max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n(G_n)|G_n) \leq \delta, \quad (21)$$

$$|\text{supp}(G_n)| \leq L. \quad (22)$$

An analogous statement holds for  $\max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n(G_n)|G_n)$  replaced by  $\max_{\gamma \in \text{supp}(G_n)} \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n(\gamma))$ .

### C. Model robustness and continuity

Here we study the continuity of the correlated random coding secrecy capacity function in the channel. Continuity is an important property of a capacity function, a fact which is sometimes overlooked because single-letter formulas usually are obviously continuous. The question becomes non-trivial in the case of a multi-letter capacity formula like  $R_S^*(\mathfrak{W}, \mathfrak{V})$ .

Suppose the capacity function were not continuous and assume that one estimates a channel which is close to a point of discontinuity. Then this channel has to be estimated to a precision which might be higher than achievable in the estimation process, or even higher than a computer can handle with reasonable effort. Otherwise, the capacity

expression obtained from the formula is next to useless for this particular channel, as all of its values in the neighbourhood of the estimated channel could be the correct one, and this range of possible values could take on arbitrary form. From this point of view, the lack of continuity of a capacity function is more dramatic than a lacking single-letter expression, because a multi-letter formula still allows an approximate calculation, whereas approximation is not possible if the capacity function is discontinuous.

We shall show that the capacity functions  $C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$  and  $C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V})$  are continuous. The argumentation relies on the fact that we have an explicit formula for these, as  $C_{S,\text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V}) = C_{S,\text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V}) = R_S^*(\mathfrak{W}, \mathfrak{V})$ . It is thus an example of the usefulness of a multi-letter formula.

Of course, the set of AVWCs with given in- and output alphabets has to be equipped with a metric in order to be able to talk about the continuity of capacity in the channel. Let  $(\mathfrak{W}, \mathfrak{V})$  and  $(\tilde{\mathfrak{W}}, \tilde{\mathfrak{V}})$  be two AVWCs with input alphabet  $\mathcal{A}$  and output alphabets  $\mathcal{B}, \mathcal{C}$  for the legitimate receiver and the eavesdropper, respectively. Denote the finite state space of  $(\mathfrak{W}, \mathfrak{V})$  by  $\mathcal{S}$  and the finite state space of  $(\tilde{\mathfrak{W}}, \tilde{\mathfrak{V}})$  by  $\tilde{\mathcal{S}}$ . We measure the distance of  $(\mathfrak{W}, \mathfrak{V})$  and  $(\tilde{\mathfrak{W}}, \tilde{\mathfrak{V}})$  by what is called the *Hausdorff distance* of two sets.

For two stochastic matrices  $W, \tilde{W} : \mathcal{A} \rightarrow \mathcal{B}$ , we define

$$\|W - \tilde{W}\|_o := \max_{a \in \mathcal{A}} \|W(\cdot | a) - \tilde{W}(\cdot | a)\|.$$

We define four asymmetric distances

$$d_{B,1}(\mathfrak{W}, \tilde{\mathfrak{W}}) := \max_{\tilde{s} \in \tilde{\mathcal{S}}} \min_{s \in \mathcal{S}} \|W_s - \tilde{W}_{\tilde{s}}\|_o,$$

$$d_{B,2}(\mathfrak{W}, \tilde{\mathfrak{W}}) := \max_{s \in \mathcal{S}} \min_{\tilde{s} \in \tilde{\mathcal{S}}} \|W_s - \tilde{W}_{\tilde{s}}\|_o,$$

and analogously define  $d_{E,1}(\mathfrak{V}, \tilde{\mathfrak{V}}), d_{E,2}(\mathfrak{V}, \tilde{\mathfrak{V}})$  by replacing  $W_s, \tilde{W}_{\tilde{s}}$  in the above definitions by  $V_s, \tilde{V}_{\tilde{s}}$ . Then the Hausdorff distance between  $(\mathfrak{W}, \mathfrak{V})$  and  $(\tilde{\mathfrak{W}}, \tilde{\mathfrak{V}})$  is defined by

$$d((\mathfrak{W}, \mathfrak{V}), (\tilde{\mathfrak{W}}, \tilde{\mathfrak{V}})) := \max\{d_{B,1}(\mathfrak{W}, \tilde{\mathfrak{W}}), d_{E,1}(\mathfrak{V}, \tilde{\mathfrak{V}}), d_{B,2}(\mathfrak{W}, \tilde{\mathfrak{W}}), d_{E,2}(\mathfrak{V}, \tilde{\mathfrak{V}})\}.$$

One checks easily that this is an actual metric on the set of finite-state AVWCs with the corresponding alphabets  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ .

Building on Theorem 6, we now state the central result concerning the continuity of the correlated random capacities.

*Theorem 9:*  $R_S^*(\mathfrak{W}, \mathfrak{V})$  is continuous in  $(\mathfrak{W}, \mathfrak{V})$  with respect to the metric  $d$ . Thus,  $C_{S,\text{ran}}(\mathfrak{W}, \mathfrak{V})$  and  $\hat{C}_{S,\text{ran}}(\mathfrak{W}, \mathfrak{V})$  are continuous functions of the channel.

The proof of this theorem only requires minor changes compared to that of [9, Theorem 2] where the continuity of the capacity of the corresponding compound wiretap channel is shown.

In contrast to the correlated random coding secrecy capacity, the uncorrelated coding secrecy capacity of AVWCs is known to be discontinuous. This was shown in [9] with a very simple example on small alphabets and a state set of no more than two elements. Hence the continuity of the correlated random coding secrecy capacity becomes even more remarkable, especially as the previous subsection IV-B has shown that only very little correlated randomness

is required to cause such a qualitative change of capacity functions. The exact characterization of the discontinuity points of the uncorrelated coding secrecy capacity  $C_S(\overline{\mathfrak{W}}, \mathfrak{V})$  is more intricate. It is discussed in depth in [21].

## V. THE COMPOUND-ARBITRARILY VARYING WIRETAP CHANNEL

To establish Theorem 6, we use Ahlswede's robustification technique [2]. It was developed to turn deterministic codes for compound channels into correlated random codes for AVCs. It has already been applied in [4] to compound and arbitrarily varying wiretap channels. The difference of this paper's approach is that the channel from sender to eavesdropper will always be arbitrarily varying. Therefore it is no longer necessary to assume the existence of a best channel to the eavesdropper.

We now formalize the idea of having a compound channel from  $\mathcal{A}$  to  $\mathcal{B}$  and an arbitrarily varying channel from  $\mathcal{A}$  to  $\mathcal{C}$ . Let  $\mathcal{R}$  be any set. For every  $r \in \mathcal{R}$ , let  $W_r : \mathcal{X} \rightarrow \mathcal{Y}$  be a stochastic matrix. Set  $W_r^n(y^n|x^n) = \prod_{i=1}^n W_r(y_i|x_i)$ . Note that here, in contrast to the AVC, the channel state remains constant over time. This defines a *compound channel*  $\overline{\mathfrak{W}} := \{W_r^n : r \in \mathcal{R}, n = 1, 2, \dots\}$ . Together with the AVC  $\mathfrak{V}$  from the previous section, we obtain the *compound-arbitrarily varying wiretap channel* (CAVWC)  $(\overline{\mathfrak{W}}, \mathfrak{V})$ .

We apply uncorrelated  $(n, J_n)$ -codes for message transmission over  $(\overline{\mathfrak{W}}, \mathfrak{V})$ . Together with  $(\overline{\mathfrak{W}}, \mathfrak{V})$ , every  $(n, J_n)$ -code defines a canonical family of random variables

$$\mathcal{F}(\mathcal{K}_n, \overline{\mathfrak{W}}, \mathfrak{V}) := \{(M^n, X^n, Y_r^n, Z_{s^n}^n, \hat{M}_r^n) : r \in \mathcal{R}, s^n \in \mathcal{S}^n\}, \quad (23)$$

where  $M^n$  and  $\hat{M}_r^n$  assume values in  $\mathcal{J}_n$ , the values of  $X^n$  lie in  $\mathcal{A}^n$ , those of  $Y_r^n$  in  $\mathcal{B}^n$  and those of  $Z_{s^n}^n$  in  $\mathcal{C}^n$  and where for any  $r \in \mathcal{R}$  and  $s^n \in \mathcal{S}^n$

$$P_{M^n X^n Y_r^n Z_{s^n}^n \hat{M}_r^n}(j, x^n, y^n, z^n, \hat{j}) = \frac{1}{J_n} E(x^n|j) W_r^n(y^n|x^n) V_{s^n}^n(z^n|x^n) \mathbb{1}_{\mathcal{D}_j}(y^n).$$

For the uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n$ , the average error is defined as

$$\bar{e}(\mathcal{K}_n) := \max_{r \in \mathcal{R}} \mathbb{P}[M^n \neq \hat{M}_r^n].$$

*Definition 10:* A nonnegative number  $R_S$  is called an *achievable secrecy rate for the CAVWC*  $(\overline{\mathfrak{W}}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n)_{n=1}^\infty$  of uncorrelated  $(n, J_n)$ -codes such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log J_n &\geq R_S, \\ \lim_{n \rightarrow \infty} \bar{e}(\mathcal{K}_n) &= 0, \\ \lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n) &= 0. \end{aligned} \quad (24)$$

The supremum of all achievable secrecy rates is called the *secrecy capacity of*  $(\overline{\mathfrak{W}}, \mathfrak{V})$  and denoted by  $C_S(\overline{\mathfrak{W}}, \mathfrak{V})$ .

We are actually interested in a stronger, permutation invariant form of secrecy. This is because we mainly consider CAVWCs as an auxiliary channel model. We would like to exploit the achievability part of a coding theorem for CAVWCs to find rates that are achievable for the AVWC by correlated random codes. This can be done using Ahlswede's robustification technique, which requires an exponential decrease of the average error and "permutation invariance" of secrecy to be defined below.

For a permutation  $\pi$  contained in the symmetric group  $\Pi_n$  of permutations of  $\{1, \dots, n\}$ , denote by  $E^\pi$  the stochastic encoder obtained from a stochastic encoder  $E$  via

$$E^\pi(x^n|j) := E(\pi^{-1}(x^n)|j). \quad (25)$$

Here,  $\pi(x^n) = (x_{\pi(1)}, \dots, x_{\pi(n)})$  for any  $x^n \in \mathcal{A}^n$ . The corresponding decoding sets are  $\mathcal{D}_j^\pi := \{\pi(y^n) : y^n \in \mathcal{D}_j\}$ . This family of codes together with  $(\overline{\mathfrak{M}}, \mathfrak{V})$  induces a canonical *permutation-invariant* family of random variables

$$\mathcal{F}(\mathcal{K}_n, \overline{\mathfrak{M}}, \mathfrak{V}, \Pi_n) := \{(M^n, X^n(\pi), Y_r^n(\pi), Z_{s^n}^n(\pi), \hat{M}_r^n(\pi)) : r \in \mathcal{R}, s^n \in \mathcal{S}^n, \pi \in \Pi_n\}, \quad (26)$$

where  $M^n$  and  $\hat{M}_r^n(\pi)$  assume values in  $\mathcal{J}_n$ , the values of  $X^n(\pi)$  lie in  $\mathcal{A}^n$ , those of  $Y_r^n(\pi)$  in  $\mathcal{B}^n$  and those of  $Z_{s^n}^n(\pi)$  in  $\mathcal{C}^n$  and where for any  $r \in \mathcal{R}$  and  $s^n \in \mathcal{S}^n$  and  $\pi \in \Pi_n$

$$P_{M^n X^n(\pi) Y_r^n(\pi) Z_{s^n}^n(\pi) \hat{M}_r^n(\pi)}(j, x^n, y^n, z^n, \hat{j}) = \frac{1}{J_n} E^\pi(x^n|j) W_r^n(y^n|x^n) V_{s^n}^n(z^n|x^n) \mathbb{1}_{\mathcal{D}_j^\pi}(y^n).$$

For every permutation, we have  $\mathbb{P}[M^n \neq \hat{M}^n(\pi)] = \mathbb{P}[M^n \neq \hat{M}^n(\text{id})]$ , where  $\text{id}$  denoted the identity permutation.

Thus also in the permutation-invariant setting, we can still just write  $\bar{e}(\mathcal{K}_n)$  for the average error of  $\mathcal{K}_n$ .

*Definition 11:* A nonnegative number  $R_S$  is called an *achievable permutation invariant secrecy rate for the CAVWC*  $(\overline{\mathfrak{M}}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n)_{n=1}^\infty$  of uncorrelated  $(n, J_n)$ -codes and a  $\beta > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log J_n \geq R_S, \quad (27)$$

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \bar{e}(\mathcal{K}_n) \geq \beta, \quad (28)$$

$$\lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} \max_{\pi \in \Pi_n} I(M^n \wedge Z_{s^n}^n(\pi)) = 0. \quad (29)$$

The supremum of all achievable permutation invariant secrecy rates is called the *permutation invariant secrecy capacity of*  $(\overline{\mathfrak{M}}, \mathfrak{V})$  and denoted by  $C_S^{\pi\text{-inv}}(\overline{\mathfrak{M}}, \mathfrak{V})$ .

*Theorem 12:* The permutation invariant secrecy capacity  $C_S^{\pi\text{-inv}}(\overline{\mathfrak{M}}, \mathfrak{V})$  and the secrecy capacity  $C_S(\overline{\mathfrak{M}}, \mathfrak{V})$  of the CAVWC  $(\overline{\mathfrak{M}}, \mathfrak{V})$  both equal

$$R_S^*(\overline{\mathfrak{M}}, \mathfrak{V}) := \lim_{k \rightarrow \infty} \frac{1}{k} \sup_{\{\bar{U}, \bar{X}^k, \bar{Y}_r^k, \bar{Z}_{s^k}^k\}} \left( \min_{r \in \mathcal{R}} I(\bar{U} \wedge \bar{Y}_r^k) - \max_{s^k \in \mathcal{S}^k} I(\bar{U} \wedge \bar{Z}_{s^k}^k) \right),$$

where the supremum is over the set of families of random variables

$$\{\bar{U}, \bar{X}^k, \bar{Y}_r^k, \bar{Z}_{s^k}^k : r \in \mathcal{R}, s^k \in \mathcal{S}^k\}$$

satisfying that  $\bar{U}$  assumes values in a finite subset of the integers, the values of  $\bar{X}^k$  lie in  $\mathcal{A}^k$ , those of  $\bar{Y}_r^k$  in  $\mathcal{B}^k$ , those of  $\bar{Z}_{s^k}^k$  in  $\mathcal{C}^k$ , and such that for every  $r \in \mathcal{R}$  and  $s^k \in \mathcal{S}^k$ ,

$$P_{\bar{U} \bar{X}^k \bar{Y}_r^k \bar{Z}_{s^k}^k}(u, x^k, y^k, z^k) = P_{\bar{U}}(u) P_{\bar{X}^k | \bar{U}}(x^k | u) W_r^k(y^k | x^k) V_{s^k}^k(z^k | x^k).$$

$P_{\bar{U}}$  and  $P_{\bar{X}^k | \bar{U}}$  may be arbitrary probability distributions and stochastic matrices, respectively.

Remarks 7-1), 7-2), 7-4) and 7-5) apply here as well.

## VI. ACHIEVABILITY PART OF THE PROOF OF THEOREM 12

### A. Reduction

- As  $C_S^{\pi\text{-inv}}(\overline{\mathfrak{W}}, \mathfrak{V}) \leq C_S(\overline{\mathfrak{W}}, \mathfrak{V})$ , it is sufficient to show that  $R_S^*(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable permutation invariant secrecy rate for  $(\overline{\mathfrak{W}}, \mathfrak{V})$ .
- Call  $R_S \geq 0$  an *achievable secrecy rate with exponentially decreasing error for the CAVWC*  $(\overline{\mathfrak{W}}, \mathfrak{V})$  if there exists a sequence  $(\mathcal{K}_n)_{n=1}^\infty$  of uncorrelated  $(n, J_n)$ -codes and a  $\beta > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log J_n \geq R_S, \quad (30)$$

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \bar{e}(\mathcal{K}_n) \geq \beta, \quad (31)$$

$$\lim_{n \rightarrow \infty} \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n) = 0, \quad (32)$$

where  $M^n$  and the  $Z_{s^n}^n$  are the corresponding elements of  $\mathcal{F}(\mathcal{K}_n, \overline{\mathfrak{W}}, \mathfrak{V})$ . It is sufficient to prove that  $R_S^*(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable secrecy rate with exponentially decreasing error for  $(\overline{\mathfrak{W}}, \mathfrak{V})$ . This is due to the following lemma.

*Lemma 13:* Let  $\mathcal{K}_n$  be an uncorrelated  $(n, J_n)$ -code with stochastic encoder  $E$ . Let  $M^n$  be the canonical message random variable and  $\{Z_{s^n}^n(\pi) : s^n \in \mathcal{S}^n, \pi \in \Pi_n\}$  the family of canonical eavesdropper output random variables from  $\mathcal{F}(\mathcal{K}_n, \overline{\mathfrak{W}}, \mathfrak{V}, \Pi_n)$ . Let  $\text{id}$  be the identity permutation mapping each element of  $\{1, \dots, n\}$  to itself. If there exists an  $\varepsilon > 0$  such that

$$\max_{s^n} I(M^n \wedge Z_{s^n}^n(\text{id})) \leq \varepsilon, \quad (33)$$

then

$$\max_{\pi \in \Pi_n} \max_{s^n} I(M^n \wedge Z_{s^n}^n(\pi)) \leq \varepsilon. \quad (34)$$

Lemma 13 is proved in Appendix B and bases on the fact that  $P_{M^n \pi(Z_{s^n}^n(\text{id}))} = P_{M^n Z_{\pi(s^n)}^n(\pi)}$ .

- $R_S^*(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable secrecy rate with exponentially decreasing error if, for every CAVWC  $(\overline{\mathfrak{W}}, \mathfrak{V})$ , the rate

$$R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V}) := \max_{\{\bar{X}, \bar{Y}_r, \bar{Z}_q\}} \left( \min_{r \in \mathcal{R}} I(\bar{X} \wedge \bar{Y}_r) - \max_{q \in \mathcal{P}(\mathcal{S})} I(\bar{X} \wedge \bar{Z}_q) \right), \quad (35)$$

is an achievable secrecy rate with exponentially decreasing error for  $(\overline{\mathfrak{W}}, \mathfrak{V})$ , where the maximum is over families of random variables  $\{\bar{X}, \bar{Y}_r, \bar{Z}_q : r \in \mathcal{R}, q \in \mathcal{P}(\mathcal{S})\}$ , with  $\bar{X}$  an arbitrary random variable assuming values in  $\mathcal{A}$ , the values of  $\bar{Y}_r$  in  $\mathcal{B}$ , those of  $\bar{Z}_q$  in  $\mathcal{C}$ , and

$$P_{\bar{X} \bar{Y}_r \bar{Z}_q}(x, y, z) = P_{\bar{X}}(x) W_r(y|x) \left( \sum_{s \in \mathcal{S}} q(s) V_s(z|x) \right).$$

This is proved using a standard channel prefixing argument, see Appendix C.

### B. $R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V})$ is an achievable secrecy rate with exponentially decreasing error

The proof that  $R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable secrecy rate with exponentially decreasing error for  $(\overline{\mathfrak{W}}, \mathfrak{V})$  follows a random coding strategy. The random codewords are chosen as follows. Fix a blocklength  $n$  and a family  $\{\bar{X}, \bar{Y}_r, \bar{Z}_q :$

$r \in \mathcal{R}, q \in \mathcal{P}(S)\}$  as in the definition of  $R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V})$ . For arbitrary  $\tau > 0$ , set<sup>2</sup>

$$\begin{aligned} J_n &:= \left\lfloor \exp \left\{ n \left( \min_{r \in \mathcal{R}} I(\bar{X} \wedge \bar{Y}_r) - \max_{q \in \mathcal{P}(S)} I(\bar{X} \wedge \bar{Z}_q) - \tau \right) \right\} \right\rfloor, \\ L_n &:= \left\lfloor \exp \left\{ n \max_{q \in \mathcal{P}(S)} I(\bar{X} \wedge \bar{Y}_r) + \frac{\tau}{4} \right\} \right\rfloor. \end{aligned} \quad (36)$$

and define  $\mathcal{J}_n = \{1, \dots, J_n\}$  and  $\mathcal{L}_n := \{1, \dots, L_n\}$ . Further, for some  $\delta > 0$  to be chosen later, we define a family  $\mathcal{X} := \{X_{jl} : j \in \mathcal{J}_n, l \in \mathcal{L}_n\}$  of random codewords in  $\mathcal{X}^n$  with distribution

$$\mathbb{P}[X_{jl} = x^n] := P'(x^n) := \frac{P_{\bar{X}}^n(x^n)}{P_{\bar{X}}^n(\mathcal{T}_{\bar{X}, \delta}^n)} \mathbb{1}_{\mathcal{T}_{\bar{X}, \delta}^n}(x^n).$$

Via  $\mathcal{X}$ , we obtain a randomly selected stochastic encoder

$$E^{\mathcal{X}}(x^n | j) := \frac{1}{L_n} \sum_{l=1}^{L_n} \mathbb{1}_{\{X_{jl}\}}(x^n). \quad (37)$$

1) *Reliability*: With high probability, a realization of  $E^{\mathcal{X}}$  determines an uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n^{\text{ran}}$  for the compound channel  $\overline{\mathfrak{W}}$  with exponentially small average error.

*Lemma 14*: For sufficiently small  $\delta > 0$  there exists a  $\tau_6 > 0$  such that, if  $n$  is sufficiently large, there exist decoding sets  $\{\mathcal{D}_j^{\mathcal{X}} : j \in \mathcal{J}_n\}$  depending on  $\mathcal{X}$  such that the event

$$\iota_3 := \left\{ \sup_{r \in \mathcal{R}} \frac{1}{J_n} \sum_j \sum_{x^n} E^{\mathcal{X}}(x^n | j) W_r^n((D_j^{\mathcal{X}})^c | x^n) \leq 2^{-n\tau_6} \right\}$$

has probability at least  $1 - 2^{-n\tau_6}$ .

As the probability distribution of  $\mathcal{X}$  is not completely standard, we include a proof of this lemma in Appendix E, although it does not differ much from the proof in [6]. The proof shows that the receiver can even decode the randomization index  $l$  in addition to the messages.

2) *Secrecy*:  $\mathcal{K}_n^{\mathcal{X}}$  also satisfies the secrecy condition (32) with high probability. Recall that every realization of  $\mathcal{X}$  together with the decoding sets  $\{\mathcal{D}_j^{\mathcal{X}} : j \in \mathcal{J}_n\}$  from Lemma 14 gives rise to a canonical family of random variables  $\mathcal{F}(\mathcal{K}_n^{\mathcal{X}}, \overline{\mathfrak{W}}, \mathfrak{V}) = \{M^n, X^n, Y_r^n, Z_{s^n}^n, \hat{M}_r^n : r \in \mathcal{R}, s^n \in \mathcal{S}^n\}$  as in (4). The dependence of these random variables on  $\mathcal{X}$  is suppressed in the notation.

*Lemma 15*: For  $\delta > 0$  sufficiently small, there exist  $\tau_1, \tau_2 > 0$  such that if  $n$  is large enough, there exists a family  $\{\Theta_{s^n} : s^n \in \mathcal{S}^n\}$  of finite measures on  $\mathcal{C}^n$  such that the probability of the event

$$\iota_0 := \left\{ \max_{j \in \mathcal{J}_n} \max_{s^n \in \mathcal{S}^n} \|P_{Z_{s^n}^n | M^n}(\cdot | j) - \Theta_{s^n}(\cdot)\| \leq 2^{-\tau_1 n} \right\}$$

is at least  $1 - 2^{-\tau_2 n}$ . (Note that  $P_{Z_{s^n}^n | M^n}(\cdot | j)$  is a random variable depending on  $\mathcal{X}$ .)

This lemma is proved in Appendix F.

*Corollary 2*: For  $\delta > 0$  small enough and  $n$  large enough, for the  $\tau_1, \tau_2$  from Lemma 15, the probability of the event

$$\iota'_0 := \left\{ \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n) \leq 2^{-\frac{\tau_1}{2} n} \right\}$$

<sup>2</sup>Recall that we use the convention  $\exp(x) = 2^x$ .



is at least  $1 - 2^{-\tau_2 n}$ . (Note again that the joint distribution of  $Z_{s^n}^n$  and  $M^n$  is a random variable depending on  $\mathcal{X}$ .)

Corollary 2 immediately follows from Lemma 15 and the uniform continuity of mutual information in total variation distance [11, Lemma 2.7].

3) *Synthesis of reliability and secrecy*: Lemma 14 and Corollary 2 show that the probability that  $\mathcal{K}_n^{\mathcal{X}}$  satisfies (30)-(32) is positive, so a realization satisfying (30)-(32) for  $\beta = \tau'_1$  and  $R_S = R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V}) - \tau$  must exist. As  $\tau > 0$  was arbitrary, this proves that  $R_S^\dagger(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable secrecy rate with exponentially decreasing error.

## VII. PROOF OF THE ACHIEVABILITY PART OF THEOREM 6

Here we prove that  $R_S^*(\mathfrak{W}, \mathfrak{V})$  is a lower bound to  $C_{S, \text{ran}}^{\max}(\mathfrak{W}, \mathfrak{V})$  and thus by Remark 5 also to  $C_{S, \text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$ . We apply the achievability part of Theorem 12 proved in the previous section to a special CAVWC  $(\overline{\mathfrak{W}}, \mathfrak{V})$ . Its determining compound part, the family of stochastic matrices describing communication between the sender and the legitimate receiver, is given by  $\{W_q : q \in \mathcal{P}(\mathcal{S})\}$ , where  $W_q := \sum_{s \in \mathcal{S}} W_s q(s)$ . We thus obtain  $\overline{\mathfrak{W}} = \{W_q^n : q \in \mathcal{P}(\mathcal{S}), n = 1, 2, \dots\}$ . Observe that for  $R_S^*(\mathfrak{W}, \mathfrak{V})$  defined in (12), we have

$$R_S^*(\mathfrak{W}, \mathfrak{V}) = R_S^*(\overline{\mathfrak{W}}, \mathfrak{V}).$$

Central to the proof is Ahlswede's robustification technique:

*Lemma 16 ([2]):* If a function  $f : \mathcal{S}^n \rightarrow [0, 1]$  satisfies

$$\sum_{s^n \in \mathcal{S}^n} f(s^n) q(s_1) \cdots q(s_n) \geq 1 - \varepsilon \quad (38)$$

for all  $q \in \mathcal{P}_0^n(\mathcal{S})$  and some  $\varepsilon \in [0, 1]$ , then

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} f(\pi(s^n)) \geq 1 - 3 \cdot (n+1)^{|\mathcal{S}|} \cdot \varepsilon. \quad (39)$$

Let now  $\varepsilon > 0$ . By Theorem 12 applied to the CAVWC  $(\overline{\mathfrak{W}}, \mathfrak{V})$  defined above, there exists a  $\beta > 0$  such that for sufficiently large  $n$ , there exists an uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n$  satisfying

$$\begin{aligned} \frac{1}{n} \log J_n &\geq R_S^*(\overline{\mathfrak{W}}, \mathfrak{V}) - \varepsilon = R_S^*(\mathfrak{W}, \mathfrak{V}) - \varepsilon, \\ \bar{e}(\mathcal{K}_n) &= \max_{q \in \mathcal{P}(\mathcal{S})} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{x^n \in \mathcal{A}^n} E(x^n | j) W_q^n(\mathcal{D}_j^c | x^n) \leq 2^{-n(\beta - \varepsilon)}, \end{aligned} \quad (40)$$

$$\max_{s^n \in \mathcal{S}^n} \max_{\pi \in \mathcal{S}_n} I(M^n \wedge Z_{s^n}^n(\pi)) \leq \varepsilon. \quad (41)$$

Define the function  $f$  by

$$f(s^n) := \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n \in \mathcal{A}^n} E(x^n | j) W_{s^n}^n(\mathcal{D}_j | x^n).$$

It was already noted in Remark 7-4) that for any  $q \in \mathcal{P}_0^n(\mathcal{S})$  and  $x^n \in \mathcal{A}^n$  and  $y^n \in \mathcal{B}^n$

$$\sum_{s^n} W_{s^n}^n(y^n | x^n) q(s_1) \cdots q(s_n) = W_q^n(y^n | x^n).$$

Thus by (40)

$$\begin{aligned}
\sum_{s^n \in \mathcal{S}^n} f(s^n) q(s_1) \cdots q(s_n) &= \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{s^n \in \mathcal{S}^n} \sum_{x^n \in \mathcal{A}^n} E(x^n | j) W_{s^n}^n(\mathcal{D}_j | x^n) q(s_1) \cdots q(s_n) \\
&= \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{s^n \in \mathcal{S}^n} \sum_{x^n \in \mathcal{A}^n} E(x^n | j) W_q^n(\mathcal{D}_j | x^n) \\
&\geq 1 - 2^{-n(\beta - \varepsilon)}.
\end{aligned}$$

Now we derive a correlated random  $(n, J_n)$ -code  $\mathcal{K}_n^{\text{ran}}$  from  $\mathcal{K}_n$ . Let  $E^\pi$  be given by  $E^\pi(x^n | j) := E(\pi^{-1}(x^n) | j)$  and let  $\mathcal{D}_j^\pi := \{\pi(y^n) : y^n \in \mathcal{D}_j\}$ . Further let  $G_n$  be uniformly distributed on this family indexed by  $\Pi_n$ . One has

$$\begin{aligned}
1 - e(\mathcal{K}_n^{\text{ran}}) &= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n} E^{\pi^{-1}}(x^n | j) W_{s^n}^n(\mathcal{D}_j^{\pi^{-1}} | x^n) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n} E(\pi(x^n) | j) W_{s^n}^n(\mathcal{D}_j^\pi | x^n) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n} E(x^n | j) W_{s^n}^n(\mathcal{D}_j^{\pi^{-1}} | \pi^{-1}(x^n)) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi_n} \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n} E(x^n | j) W_{\pi(s^n)}^n(\mathcal{D}_j | x^n).
\end{aligned}$$

With  $\varepsilon = 2^{-n(\beta - \varepsilon)}$ , Lemma 16 implies that the last term is lower-bounded by  $1 - (n+1)^{|S|} 2^{-n(\beta - \varepsilon)} \geq 1 - 2^{-n(\beta - 2\varepsilon)}$  for sufficiently large  $n$ . This settles the reliability properties of  $\mathcal{K}_n^{\text{ran}}$ .

The secrecy properties of  $\mathcal{K}_n^{\text{ran}}$  are immediate, as (41) implies

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} I(M^n \wedge Z_{s^n}^n(\pi)) \leq \max_{\pi \in \Pi_n} I(M^n \wedge Z_{s^n}^n(\pi)) \leq \varepsilon$$

for every  $s^n \in \mathcal{S}^n$ . Hence  $R_S^*(\overline{\mathfrak{W}}, \mathfrak{V})$  is an achievable correlated random coding maximal secrecy rate.

## VIII. THE CONVERSES

One unusual difficulty arises in the proof of the converse of Theorem 6. This difficulty consists in the fact that the common randomness prohibits a “naive” application of the data processing inequality. It is thus necessary to limit the amount of common randomness of an arbitrary correlated random code in order to overcome this difficulty. This has already been done in Lemma 8.

Let  $R_S < C_{S, \text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V})$ . From Lemma 8 we know that for every  $\varepsilon > 0$  there is an  $L = L(R_S, \varepsilon)$  such that for sufficiently large  $n$  there is a correlated random  $(n, J_n)$ -code  $\mathcal{K}_n^{\text{ran}}$  satisfying

$$\frac{1}{n} \log J_n \geq R_S - \varepsilon, \tag{42}$$

$$e(\mathcal{K}_n^{\text{ran}}) \leq \varepsilon, \tag{43}$$

$$\max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n | G_n) \leq \varepsilon, \tag{44}$$

$$|\text{supp}(G_n)| = L(R_S, \varepsilon). \tag{45}$$

By [11, Lemma 12.3], the average error incurred by any uncorrelated code  $\mathcal{K}_n$  used over the AVC  $\mathfrak{W}$  equals the average error of  $\mathcal{K}_n$  over the AVC determined by the convex hull of  $\{W_s : s \in \mathcal{S}\}$ , i. e. the AVC  $\{W_{q^n}^n : q^n \in \mathcal{P}(\mathcal{S})^n, n = 1, 2, \dots\}$ , where

$$W_{q^n}^n(y^n|x^n) := \prod_{i=1}^n \sum_{s_i \in \mathcal{S}} W_{s_i}(y_i|x_i) q_i(s_i).$$

This is a simple consequence of the fact that the average error is affine in the channel and carries over to correlated random codes. Hence (43) implies

$$\max_{q^n \in \mathcal{P}(\mathcal{S})^n} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{\gamma \in \Gamma_n} \sum_{x^n \in \mathcal{A}^n} E^\gamma(x^n|j) W_{q^n}^n((\mathcal{D}_j^\gamma)^c|x^n) P_{G_n}(\gamma) \leq \varepsilon. \quad (46)$$

From (46), one infers that the average error of  $\mathcal{K}_n^{\text{ran}}$  for transmission over the compound channel  $\overline{\mathfrak{W}}$  is upper-bounded by  $\varepsilon$  as well, i. e.

$$\max_{q \in \mathcal{P}(\mathcal{S})} \frac{1}{J_n} \sum_{j=1}^{J_n} \sum_{\gamma \in \Gamma_n} \sum_{x^n \in \mathcal{A}^n} E^\gamma(x^n|j) W_q^n((\mathcal{D}_j^\gamma)^c|x^n) P_{G_n}(\gamma) \leq \varepsilon. \quad (47)$$

Due to Fano's inequality [11, Lemma 3.8], (47) implies for every  $q \in \mathcal{P}(\mathcal{S})$

$$\begin{aligned} H(M^n|\hat{M}_q^n, G_n) &= \sum_{\gamma \in \text{supp}(G_n)} H(M^n|\hat{M}_q^n, G_n = \gamma) P_{G_n}(\gamma) \\ &\leq 1 + \sum_{\gamma \in \text{supp}(G_n)} \mathbb{P}[M^n \neq \hat{M}_q^n | G_n = \gamma] P_{G_n}(\gamma) \log J_n \\ &= 1 + \varepsilon \log J_n. \end{aligned}$$

Here the  $\hat{M}_q^n$  are the random variables from the canonical family  $\mathcal{F}(\mathcal{K}_n^{\text{ran}}, \overline{\mathfrak{W}}, \mathfrak{V})$  defined in (23). Hence the independence of  $M^n$  and  $G_n$  yields

$$\log J_n = H(M^n) = H(M^n|G_n) = I(M^n \wedge \hat{M}_q^n|G_n) + H(M^n|\hat{M}_q^n, G_n) \leq I(M^n \wedge \hat{M}_q^n|G_n) + 1 + \varepsilon \log J_n,$$

so by rearranging and taking (44) into account, we have for every  $q \in \mathcal{P}(\mathcal{S})$  and  $s^n \in \mathcal{S}^n$

$$(1 - \varepsilon) \log J_n \leq I(M^n \wedge \hat{M}_q^n|G_n) - I(M^n \wedge Z_{s^n}^n|G_n) + 1 + \varepsilon.$$

We have to get rid of  $G_n$  in some way. The only reasonable way to achieve this seems to be through the use of the convexity of the mutual information in the channel argument. But while this is a valid choice for the “secrecy term”, it is certainly invalid for the “legal” term. This is due to the fact that  $G_n$  is independent of  $M^n$ , but not of  $\hat{M}_q^n$  or  $Y_q^n$ . An application of the data processing inequality is thus only possible conditioned on  $G_n$ . It is here where the importance of Lemma 8 becomes evident: The cardinality of the support of  $G_n$  is bounded and independent of  $n$  for  $n$  sufficiently large, hence we can write

$$\begin{aligned} I(M^n \wedge \hat{M}_q^n|G_n) &= H(M^n) - H(M^n|Y_q^n, G_n) \\ &\leq H(M^n) - H(M^n|Y_q^n) + H(G_n) \\ &\leq I(M^n \wedge Y_q^n) + \log L(R_S, \varepsilon), \end{aligned}$$

where we employed the fact that  $H(S) \leq H(S, T) = H(S|T) + H(T)$ . Thus if  $n$  is sufficiently large, we obtain that

$$\begin{aligned} \frac{1}{n} \log J_n &\leq \frac{1}{n(1-\varepsilon)} \left( \min_{q \in \mathcal{P}(\mathcal{S})} I(M^n \wedge \hat{M}_q^n | G_n) - \max_{s^n \in \mathcal{S}^n} (M^n \wedge Z_{s^n}^n | G_n) + 1 + \varepsilon \right) \\ &\leq \frac{1}{n(1-\varepsilon)} \left( \min_{q \in \mathcal{P}(\mathcal{S})} I(M^n \wedge Y_q^n) - \max_{s^n \in \mathcal{S}^n} I(M^n \wedge Z_{s^n}^n) \right) + \frac{\log L(R_S, \varepsilon) + 1 + \varepsilon}{n(1-\varepsilon)}. \end{aligned} \quad (48)$$

For  $n$  sufficiently large, as  $L(R_S, \varepsilon)$  is independent of  $n$ , the second term of (48) is upper-bounded by  $\varepsilon$ . If we set  $\bar{U} := M^n$  and  $\bar{X}^n := X^n$  and  $\bar{Y}_q^n := Y_q^n$  and  $\bar{Z}_{s^n}^n := Z_{s^n}^n$ , the joint distributions

$$\begin{aligned} P_{\bar{U} \bar{X}^n \bar{Y}_q^n}(j, x^n, y^n) &= \frac{1}{J_n} \sum_{\gamma \in \Gamma_n} P_{G_n}(\gamma) E^\gamma(x^n | j) W_q^n(y^n | x^n), \\ P_{\bar{U} \bar{X}^n \bar{Z}_{s^n}^n}(j, x^n, z^n) &= \frac{1}{J_n} \sum_{\gamma \in \Gamma_n} P_{G_n}(\gamma) E^\gamma(x^n | j) V_{s^n}^n(z^n | x^n) \end{aligned}$$

have the form required in the definition of  $R_S^*(\mathfrak{W}, \mathfrak{V})$ , and the shared randomness is now completely reduced to randomness at the encoder. Thus by (42) and as  $\varepsilon$  was arbitrary, we have  $R_S \leq R_S^*(\mathfrak{W}, \mathfrak{V})$ , hence  $C_{S, \text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V}) \leq R_S^*(\mathfrak{W}, \mathfrak{V})$ , and therefore also  $C_{S, \text{ran}}^{\text{max}}(\mathfrak{W}, \mathfrak{V}) \leq R_S^*(\mathfrak{W}, \mathfrak{V})$ . This completes the proof of the converse of Theorem 6.

*Remark 17:* As the average error is affine in the channel, one can even pass to a maximum over  $\tilde{q} \in \mathcal{P}(\mathcal{S}^n)$  in (46). Skipping the reduction to  $q \in \mathcal{P}(\mathcal{S})$  in (47) and directly applying Fano's inequality, the rest of the proof can be performed as above for every  $\tilde{q} \in \mathcal{P}(\mathcal{S}^n)$  using random variables  $Y_{\tilde{q}}^n = \bar{Y}_{\tilde{q}}^n$  defined by

$$P_{Y_{\tilde{q}}^n | X^n}(y^n | x^n) = \sum_{s^n} \tilde{q}(s^n) W_{s^n}^n(y^n | x^n).$$

This shows that the right-hand side of (15) upper-bounds  $C_{S, \text{ran}}^{\text{mean}}(\mathfrak{W}, \mathfrak{V}) = R_S^*(\mathfrak{W}, \mathfrak{V})$ . Since the right-hand side of (15) trivially is a lower bound on  $R_S^*(\mathfrak{W}, \mathfrak{V})$ , as noted in Remark 7-4, we can conclude the validity of equality (15).

The converse for Theorem 12 follows the same lines. It is simpler as no common randomness has to be considered.

## IX. DISCUSSION

The main result of this paper is the correlated random coding secrecy capacity of the AVWC for the case where the eavesdropper is allowed access to the correlated randomness shared by sender and intended receiver. Applying Ahlswede's robustification technique, the main problem was solved via reduction to the secrecy capacity problem of the CAVWC, which is compound between the sender and the intended receiver and arbitrarily varying between the sender and the eavesdropper.

The secrecy capacity formula obtained in the main theorem is a multi-letter formula. Of course, this makes a direct computation impossible. On the other hand, it is not known whether a general, computable, single-letter formula exists at all. For a given AVWC, the value of the multi-letter formula can be approximated by restricting computation to a finite number of letters. An open problem not addressed in this paper is the goodness of finite-letter approximation.

However, the use of a capacity formula is much larger than just to calculate the capacity. It can be applied in the in-depth analysis of the channels in question. For example, using nothing but the capacity formula, it can be shown for discrete memoryless channels that the capacity of parallel channels is the sum of their capacities. For the AVWC, an analysis of the capacity formula shows that the correlated random coding secrecy capacity is continuous in the AVWC, which is impossible to derive a priori. This result is of great engineering importance because it ensures that small variations in the channel data cannot lead to completely different secrecy capacities. This is very reassuring, as lots of resources would otherwise have to be spent on channel estimation. In fact, the necessary precision of the channel estimate would grow without limits the closer the channel would be to a point of discontinuity of the secrecy capacity function.

Follow-up work on the AVWC correlated random coding secrecy capacity for the case that the eavesdropper has no knowledge of the correlated randomness as well as the AVWC uncorrelated coding secrecy capacity is presented in [21].

## APPENDIX A

### PROOF OF COROLLARY 1

It is obvious that the right-hand side of (18) is upper-bounded by  $R_S^*(\mathfrak{W}, \mathfrak{V})$ , see Remark 7-1). Thus it remains to show the converse relation. Let  $k$  be a positive integer and let  $\{\bar{U}, \bar{X}, \bar{Y}_{q_1}^k, \bar{Z}_{s_2}^k\}$  be a family of random variables as in the definition of  $R_S^*(\mathfrak{W}, \mathfrak{V})$ . The existence of a best channel to the eavesdropper guarantees that  $I(U \wedge Z_{s_2}^k) \leq I(U \wedge Z_{s_*}^k)$  for every  $s_2^k \in \mathcal{S}_2^k$ , where  $P_{Z_{s_*}^k|X}(z^k|x^k) = \prod_{i=1}^k V_{s_*}(z_i|x_i)$ . In particular,  $I(U \wedge Z_{s_*}^k) = \max_{s_2 \in \mathcal{S}_2} I(U \wedge Z_{s_2}^k)$ . Therefore

$$\begin{aligned} \frac{1}{k} \left( \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} I(\bar{U} \wedge \bar{Y}_{q_1}^k) - \max_{s_2^k \in \mathcal{S}_2^k} I(\bar{U} \wedge \bar{Z}_{s_2^k}^k) \right) &= \frac{1}{k} \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} \left( I(\bar{U} \wedge \bar{Y}_{q_1}^k) - I(\bar{U} \wedge \bar{Z}_{s_*}^k) \right) \\ &\leq \frac{1}{k} \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} I(\bar{U} \wedge \bar{Y}_{q_1}^k | \bar{Z}_{s_*}^k), \end{aligned} \quad (49)$$

where strong degradedness was applied in (49). In a similar fashion as in the derivation of (23)-(26) in [20], one can rewrite the right-hand side of (49) as  $I(\bar{X}^* \wedge \bar{Y}_{q_1}^* | \bar{Z}_{s_*}^*)$ , where  $\bar{X}^*$  is a random variable on  $\mathcal{A}$  and the distributions of  $\bar{Y}_{q_1}^*$  and  $\bar{Z}_{s_*}^*$  satisfy  $P_{\bar{Y}_{q_1}^*|\bar{X}^*} = W_{q_1}$  and  $P_{\bar{Z}_{s_*}^*|\bar{X}^*} = V_{s_*}$ . Again using the strong degradedness of  $(\mathfrak{W}, \mathfrak{V})$  and the existence of a best channel to the eavesdropper and defining  $\bar{Z}_{s_2}^*$  by its conditional distribution  $P_{\bar{Z}_{s_2}^*|\bar{X}^*} = V_{s_2}$  for every  $s_2 \in \mathcal{S}_2$ , one obtains

$$\begin{aligned} \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} I(\bar{X}^* \wedge \bar{Y}_{q_1}^* | \bar{Z}_{s_*}^*) &\leq \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} \left( I(\bar{X}^* \wedge \bar{Y}_{q_1}^*) - I(\bar{X}^* \wedge \bar{Z}_{s_*}^*) \right) \\ &= \min_{q_1 \in \mathcal{P}(\mathcal{S}_1)} I(\bar{X}^* \wedge \bar{Y}_{q_1}^*) - \max_{s_2 \in \mathcal{S}_2} I(\bar{X}^* \wedge \bar{Z}_{s_2}^*). \end{aligned}$$

Inserting this in the definition of  $R_S^*(\mathfrak{W}, \mathfrak{V})$  shows that  $R_S^*(\mathfrak{W}, \mathfrak{V})$  is upper-bounded by the right-hand side of (18), thus proving that (18) indeed is an equality. This proves Corollary 1.

## APPENDIX B

## PROOF OF LEMMA 13

Assume  $\mathcal{K}_n$  satisfies (33) and has stochastic encoder  $E$ . Recall that  $E^\pi$  is defined by  $E^\pi(x^n|j) := E(\pi^{-1}(x^n)|j)$ . The random variables below are from the canonical permutation-invariant family  $\mathcal{F}(\mathcal{K}_n, \mathfrak{W}, \mathfrak{V}, \Pi_n)$ .

*Lemma 18:* For every  $\pi \in \Pi_n$ , we have  $P_{M_n \pi(Z_{s^n}^n(\text{id}))} = P_{M_n Z_{\pi(s^n)}^n(\pi)}$ .

*Proof:* Let  $j \in \mathcal{J}_n$  and  $z^n \in \mathcal{C}^n$ . Then

$$\begin{aligned} \mathbb{P}[M_n = j, \pi(Z_{s^n}^n(\text{id})) = z^n] &= \mathbb{P}[M_n = j, Z_{s^n}^n(\text{id}) = \pi^{-1}(z^n)] \\ &= \frac{1}{J_n} \sum_{x^n} E(x^n|j) V_{s^n}^n(\pi^{-1}(z^n)|x^n) \\ &= \frac{1}{J_n} \sum_{x^n} E(\pi^{-1}(x^n)|j) V_{s^n}^n(\pi^{-1}(z^n)|\pi^{-1}(x^n)) \\ &= \frac{1}{J_n} \sum_{x^n} E^\pi(x^n|j) V_{\pi(s^n)}^n(z^n|x^n) \\ &= \mathbb{P}[M_n = j, Z_{\pi(s^n)}^n(\pi) = z^n]. \end{aligned}$$

■

Now assume that (33) holds. Then

$$\begin{aligned} \max_{\pi \in \Pi_n} \max_{s^n} I(M_n \wedge Z_{s^n}^n(\pi)) &= \max_{\pi \in \Pi_n} \max_{s^n} I(M_n \wedge Z_{\pi(s^n)}^n(\pi)) \\ &\stackrel{(i)}{=} \max_{\pi \in \Pi_n} \max_{s^n} I(M_n \wedge \pi(Z_{s^n}^n(\text{id}))) \\ &\stackrel{(ii)}{\leq} \max_{s^n} I(M_n \wedge Z_{s^n}^n(\text{id})) \\ &\leq \varepsilon \end{aligned}$$

where Lemma 18 was applied in (i) and the data processing inequality in (ii). Thus (33) implies (34).

## APPENDIX C

## CHANNEL PREFIXING

Assume for any CAVWC  $(\overline{\mathfrak{W}}, \tilde{\mathfrak{V}})$  that  $R_S^{\dagger}(\overline{\mathfrak{W}}, \tilde{\mathfrak{V}})$  is achievable with exponentially decreasing error for  $(\overline{\mathfrak{W}}, \tilde{\mathfrak{V}})$ . We have to show that then for a given CAVWC  $(\overline{\mathfrak{W}}, \mathfrak{V})$ ,  $R_S^*(\overline{\mathfrak{W}}, \mathfrak{V})$  also is an achievable rate with exponentially decreasing error for  $(\overline{\mathfrak{W}}, \mathfrak{V})$ . Choose a positive integer  $k$ , a finite subset  $\mathcal{U}$  of the integers, and a stochastic matrix  $T : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{A}^k)$ . For every  $r \in \mathcal{R}$  and  $s^k \in \mathcal{S}^k$ , this induces stochastic matrices  $\tilde{W}_r : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{B}^k)$  and  $\tilde{V}_{s^k} : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{C}^k)$  defined by

$$\begin{aligned} \tilde{W}_r(y^k|u) &:= \sum_{x^k} T(x^k|u) W_r^k(y^k|x^k), \\ \tilde{V}_{s^k}(y^k|u) &:= \sum_{x^k} T(x^k|u) V_{s^k}^k(z^k|x^k). \end{aligned}$$

This induces families

$$\begin{aligned}\tilde{\mathcal{W}} &:= \{\tilde{W}_r^n : r \in \mathcal{R}, n = 1, 2, \dots\}, \\ \tilde{\mathcal{Y}} &:= \{\tilde{V}_{s^{kn}}^n : s^{kn} \in (\mathcal{S}^k)^n, n = 1, 2, \dots\},\end{aligned}$$

and hence a CAVWC denoted by  $(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$ . The compound part of this channel also has  $\mathcal{R}$  as its state set, the state set of the eavesdropper channel equals  $\mathcal{S}^k$ . By assumption,  $R_S^\dagger(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$  is an achievable rate with exponentially decreasing error for  $(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$ . Thus there exists a  $\beta > 0$  such that for every  $\varepsilon > 0$  and sufficiently large  $n$ , one obtains an  $(n, J_n)$ -code  $\tilde{\mathcal{K}}_n$  for  $(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$  with canonical random family  $\mathcal{F}(\tilde{\mathcal{K}}_n, \tilde{\mathcal{W}}, \tilde{\mathcal{Y}}) = \{\tilde{M}^n, \tilde{U}^n, \tilde{Y}_r^{kn}, \tilde{Z}_{s^{kn}}^{kn}, \tilde{M}_r^n : r \in \mathcal{R}, s^{kn} \in (\mathcal{S}^k)^n\}$  satisfying

$$\frac{1}{n} \log J_n \geq R_S^\dagger(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}}) - \varepsilon, \quad (50)$$

$$-\frac{1}{n} \log \bar{e}(\tilde{\mathcal{K}}_n) \geq \beta - \varepsilon, \quad (51)$$

$$\max_{s^{kn} \in (\mathcal{S}^k)^n} I(\tilde{M}^n \wedge \tilde{Z}_{s^{kn}}^{kn}) \leq \varepsilon. \quad (52)$$

Now define the stochastic encoder  $E : \mathcal{J}_n \rightarrow \mathcal{P}(\mathcal{A}^{kn})$  through

$$E(x^{kn}|j) := \sum_{u^n \in \mathcal{U}^n} E^*(u^n|j) T^n(x^{kn}|u^n).$$

Together with the decoding sets  $\mathcal{D}_j^*$  considered as sets  $\mathcal{D}_j \subset \mathcal{B}^{kn}$ , this defines an uncorrelated  $(kn, J_n)$ -code  $\mathcal{K}_{kn}$  for the CAVWC  $(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$ . Observe that, if  $\mathcal{F}(\mathcal{K}_{kn}, \tilde{\mathcal{W}}, \tilde{\mathcal{Y}}) = \{M^n, X^n, Y_r^{kn}, Z_{s^{kn}}^{kn}, M_r^n : r \in \mathcal{R}, s^{kn} \in \mathcal{S}^{kn}\}$  is the canonical random family of  $\mathcal{K}_{kn}$ , then for every  $r \in \mathcal{R}_n$  and  $s^{kn}$  regarded either as an element of  $\mathcal{S}^{kn}$  or  $(\mathcal{S}^k)^n$ , the joint probability of  $(M^n, Y_r^{kn}, Z_{s^{kn}}^{kn}, \hat{M}_r^{kn})$  equals that of  $(\tilde{M}^n, \tilde{Y}_r^{kn}, \tilde{Z}_{s^{kn}}^{kn}, \tilde{M}_r^n)$ .

It immediately follows that

$$\begin{aligned}\frac{1}{kn} \log J_n &\geq \frac{1}{k} R_S^\dagger(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}}) - \frac{\varepsilon}{k}, \\ -\frac{1}{kn} \bar{e}(\mathcal{K}_{kn}) &\geq \frac{\beta - \varepsilon}{k}, \\ \max_{s^{kn} \in \mathcal{S}^{kn}} I(M^n \wedge Z_{s^{kn}}^{kn}) &\leq \varepsilon.\end{aligned}$$

Thus after optimization over  $T$  and  $k$ , it follows that  $R_S^*(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$  is an achievable secrecy rate with exponentially decreasing error for  $(\tilde{\mathcal{W}}, \tilde{\mathcal{Y}})$ .

## APPENDIX D

### TYPES AND TYPICAL SEQUENCES

The proofs require some facts about types and typical sequences. For reference, we include them here.  $\mathcal{A}, \mathcal{B}$  and  $W, \tilde{W}$  are generic sets/stochastic matrices.



*Lemma 19:* Let  $\bar{X}$  be an  $\mathcal{A}$ -valued random variable and let  $x^n \in \mathcal{T}_{\bar{X},\delta}^n$ . Further let  $W : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{S})$ . Then for any  $\mathcal{B}$ -valued random variable  $\bar{Y}$  with  $P_{\bar{Y}|\bar{X}} = W$ ,

$$|\mathcal{T}_{\bar{Y},\delta}^n| \leq \exp\{n(H(\bar{Y}) + f_1(\delta))\},$$

$$W^n(y^n|x^n) \leq \exp\{-n(H(\bar{Y}|\bar{X}) - f_2(\delta))\} \quad \text{for all } y^n \in \mathcal{T}_{\bar{Y}|\bar{X},\delta}^n(x^n)$$

with universal  $f_1(\delta), f_2(\delta) > 0$  satisfying  $\lim_{\delta \rightarrow 0} f_1(\delta) = \lim_{\delta \rightarrow 0} f_2(\delta) = 0$ .

*Lemma 20:* Let  $\delta > 0$ . Let  $(\bar{X}, \bar{Y})$  assume values in  $\mathcal{A} \times \mathcal{B}$  such that  $P_{\bar{Y}|\bar{X}} = W$ , for some  $W : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{B})$ , and let  $x^n \in \mathcal{A}^n$ . There exist a universal  $c' > 0$  and an  $n_0 = n_0(|\mathcal{A}|, |\mathcal{B}|, \delta) \geq 1$  such that for  $n \geq n_0$

$$P_{\bar{X}}^n(\mathcal{T}_{\bar{X},\delta}^n) \geq 1 - 2^{-nc'\delta^2},$$

$$W^n(\mathcal{T}_{\bar{Y}|\bar{X},\delta}^n(x^n)|x^n) \geq 1 - 2^{-nc'\delta^2}.$$

*Lemma 21:* The cardinality of  $\mathcal{P}_0^n(\mathcal{S})$  is upper-bounded by  $(n+1)^{|\mathcal{S}|}$ .

The proofs of Lemmas 19-21 can be found in e.g. [11]. A proof of the next lemma can be found in [5].

*Lemma 22:* Let  $(\bar{X}, \bar{Y})$  and  $(\bar{X}', \bar{Y}')$  two pairs of  $\mathcal{A} \times \mathcal{B}$ -valued random variables. Then for sufficiently small  $\delta > 0$  and any positive integer  $n$ ,

$$P_{\bar{Y}}^n(\mathcal{T}_{\bar{Y}'|\bar{X}',\delta}^n(x^n)) \leq (n+1)^{|\mathcal{A}||\mathcal{B}|} \exp\{-n(I(\bar{X}' \wedge \bar{Y}') - f_3(\delta))\} \quad (53)$$

for all  $\tilde{x}^n \in \mathcal{T}_{\bar{X}',\delta}^n$  holds for a universal  $f_3(\delta) > 0$  with  $\lim_{n \rightarrow \infty} f_3(\delta) = 0$ .

Note that the right-hand side of (53) does not depend on  $(\bar{X}, \bar{Y})$ , so one might wonder how sharp this bound is. But we will apply the lemma in a case where  $\bar{X} = \bar{X}'$  and where  $P_{\bar{Y}|\bar{X}}$  and  $P_{\bar{Y}'|\bar{X}'}$  may be close (see Appendix E). Thus it turns out to give the correct upper bound.

## APPENDIX E

### PROOF OF LEMMA 14

The fact that the probability of  $\bar{e}(\mathcal{K}_n^{\mathcal{X}})$  being small is large is well-known in principle, cf. [11]. As our choice of codewords does not quite follow the standard approach and we use stochastic encoders, we present the proof nonetheless. We start with a lemma which assumes a finite state set for  $\overline{\mathcal{W}}$  and actually shows that the sender can also reliably decode the randomization index with high probability.

*Lemma 23:* Let  $\mathcal{R}' \subset \mathcal{R}$  be finite. With

$$\hat{\mathcal{D}}_{jl}^{\mathcal{X}} := \bigcup_{r \in \mathcal{R}} \mathcal{T}_{\bar{Y}_r|\bar{X},\delta}^n(X_{jl}),$$

define

$$\tilde{\mathcal{D}}_{jl}^{\mathcal{X}} := \hat{\mathcal{D}}_{jl}^{\mathcal{X}} \cap \left( \bigcup_{(j',l') \in \mathcal{J}_n \times \mathcal{L}_n \setminus \{(j,l)\}} \hat{\mathcal{D}}_{j'l'}^{\mathcal{X}} \right)^c.$$

In order for these decoding sets to cover the complete output space, we assume without loss of generality that  $\tilde{\mathcal{D}}_{11}^{\mathcal{X}}$  contains all  $y^n \in \mathcal{B}^n$  not assigned to any message so far. This does not increase the average error. The  $\tilde{\mathcal{D}}_{jl}^{\mathcal{X}}$  are

pairwise disjoint  $((j, l) \in \mathcal{J}_n \times \mathcal{L}_n)$ . For  $\tau \geq \tau_0(\delta)$ , with  $\tau_0(\delta) \rightarrow 0$  as  $\delta > 0$ , there exists an  $a = a(\tau, \delta) > 0$  such that the event

$$\tilde{t}_3 := \left\{ \max_{r \in \mathcal{R}'} \frac{1}{J_n L_n} \sum_{(j,l) \in \mathcal{J}_n \times \mathcal{L}_n} W_r^n((\tilde{\mathcal{D}}_{jl}^{\mathcal{X}})^c | X_{jl}) \leq 2^{-na} \right\}$$

has probability at least  $1 - 2^{-na}$ .

*Proof:* The disjointness of the decoding sets is obvious. We first show an upper bound on the mean error incurred by  $\mathcal{K}_n^{\mathcal{X}}$  for given state  $r \in \mathcal{R}'$ . More precisely, setting

$$e_r(\mathcal{K}_n^{\mathcal{X}}) := \frac{1}{J_n L_n} \sum_{j=1}^{J_n} \sum_{l=1}^{L_n} W_r^n((\tilde{\mathcal{D}}_{jl}^{\mathcal{X}})^c | X_{jl}),$$

we claim

$$\mathbb{E}[e_r(\mathcal{K}_n^{\mathcal{X}})] \leq 2^{-na'} \quad (54)$$

for some  $a' = a'(\tau, \delta) > 0$  and for  $n$  sufficiently large. The left-hand side of (54) equals

$$\begin{aligned} & \mathbb{E} \left[ W_r^n((\tilde{\mathcal{D}}_{11}^{\mathcal{X}})^c | X_{11}) \right] \\ & \leq \mathbb{E} \left[ W_r^n((\hat{\mathcal{D}}_{11}^{\mathcal{X}})^c | X_{11}) \right] \end{aligned} \quad (55)$$

$$+ \sum_{\substack{(j,l) \in \mathcal{J}_n \times \mathcal{L}_n: \\ (j,l) \neq (1,1)}} \mathbb{E} \left[ W_r^n(\hat{\mathcal{D}}_{jl}^{\mathcal{X}} | X_{11}) \right]. \quad (56)$$

For (55), we have

$$\mathbb{E} \left[ W_r^n((\hat{\mathcal{D}}_{11}^{\mathcal{X}})^c | X_{11}) \right] \leq \mathbb{E} \left[ W_r^n((\mathcal{T}_{\bar{Y}_r | \bar{X}, \delta}^n(X_{11}))^c | X_{11}) \right],$$

which by Lemma 20 is upper-bounded by  $2^{-nc'\delta^2}$  for  $n$  sufficiently large. Thus (55) is upper-bounded by the same number. For each of the terms in (56), we obtain

$$\mathbb{E} \left[ W_r^n(\hat{\mathcal{D}}_{jl}^{\mathcal{X}} | X_{11}) \right] \leq \sum_{r' \in \mathcal{R}'} \mathbb{E} \left[ W_r^n(\mathcal{T}_{\bar{Y}_{r'} | \bar{X}, \delta}^n(X_{jl}) | X_{11}) \right].$$

For sufficiently large  $n$ , the terms on the right-hand side can be written (recall that  $(j, l) \neq (1, 1)$ )

$$\begin{aligned} & \mathbb{E} \left[ W_r^n(\mathcal{T}_{\bar{Y}_{r'} | \bar{X}, \delta}^n(X_{jl}) | X_{11}) \right] \\ & = \sum_{x^n, \tilde{x}^n \in \mathcal{T}_{\bar{X}, \delta}^n} W_r^n(\mathcal{T}_{\bar{Y}_{r'} | \bar{X}, \delta}^n(\tilde{x}^n) | x^n) P'(x^n) P'(\tilde{x}^n) \\ & \stackrel{(i)}{\leq} (1 - 2^{-nc'\delta})^{-2} \sum_{\tilde{x}^n \in \mathcal{T}_{\bar{X}, \delta}^n} P_{\bar{Y}_r}^n(\mathcal{T}_{\bar{Y}_{r'} | \bar{X}, \delta}^n(\tilde{x}^n)) P_{\bar{X}}^n(\tilde{x}^n), \end{aligned} \quad (57)$$

where we used the definition of  $P'$  and Lemma 20 in (i). By Lemma 22,

$$P_{\bar{Y}_r}^n(\mathcal{T}_{\bar{Y}_{r'} | \bar{X}, \delta}^n(\tilde{x}^n)) \leq (n+1)^{|\mathcal{A}||\mathcal{B}|} 2^{-n(I(\bar{X} \wedge \bar{Y}_{r'}) - f_3(\delta))}.$$

This immediately gives

$$(57) \leq (1 - 2^{-nc'\delta})^{-2} (n+1)^{|\mathcal{A}||\mathcal{B}|} 2^{-n(I(\bar{X} \wedge \bar{Y}_{r'}) - f_3(\delta))},$$

and we can upper-bound (56) by

$$|\mathcal{R}'|J_nL_n \exp\left\{-n\left(\min_{r' \in \mathcal{R}'} I(\bar{X} \wedge \bar{Y}_{r'}) - 2f_3(\delta)\right)\right\}.$$

If one chooses  $\delta$  so small that  $\tau \geq 4f_3(\delta) > 0$ , this tends to 0 exponentially. Combining the bounds on (55) and (56), we thus obtain (54) for some appropriate  $a' > 0$ .

Using the Markov inequality and setting  $a := a'/3$ , we obtain from (54)

$$\begin{aligned} \mathbb{P}\left[\bigcap_{r \in \mathcal{R}'} \{e_r(\mathcal{K}_n^{\mathcal{X}}) \leq 2^{-na}\}\right] &\geq 1 - \sum_{r \in \mathcal{R}'} \mathbb{P}[e_r(\mathcal{K}_n^{\mathcal{X}}) > 2^{-na}] \\ &\geq 1 - 2^{na} \sum_{r \in \mathcal{R}'} \mathbb{E}[e_r(\mathcal{K}_n^{\mathcal{X}})] \geq 1 - |\mathcal{R}'|2^{na}2^{-3na} \geq 1 - 2^{-na} \end{aligned}$$

for sufficiently large  $n$ . Thus the probability that  $\max_{r \in \mathcal{R}'} e_r(\mathcal{K}_n^{\mathcal{X}}) \leq 2^{-na}$  is lower-bounded by  $1 - 2^{-na}$ . This completes the proof.  $\blacksquare$

We now appeal to the approximation argument of [6], from which we conclude that the same decoding sets induce an exponentially decreasing average error for the complete state set  $\mathcal{R}$  with the same probability lower-bounded by  $1 - 2^{-na}$ . This is still true for a non-stochastic encoder, the randomization index can still be decoded.

Now recall the definition of  $E^{\mathcal{X}}$ . Together with the decoding sets

$$\mathcal{D}_j^{\mathcal{X}} := \bigcup_{l \in \mathcal{L}_n} \tilde{D}_{jl},$$

for  $j \in \mathcal{J}_n$ , this defines a randomly chosen uncorrelated  $(n, J_n)$ -code  $\mathcal{K}_n^{\mathcal{X}}$ . Note that

$$\begin{aligned} \frac{1}{J_n} \sum_{j \in \mathcal{J}_n} \sum_{x^n} E^{\mathcal{X}}(x^n | j) W^n((\mathcal{D}_j^{\mathcal{X}})^c | x^n) &= \frac{1}{J_n L_n} \sum_{j \in \mathcal{J}_n} \sum_{l \in \mathcal{L}_n} W^n((\mathcal{D}_j^{\mathcal{X}})^c | X_{jl}) \\ &\leq \frac{1}{J_n L_n} \sum_{j \in \mathcal{J}_n} \sum_{l \in \mathcal{L}_n} W^n((\hat{\mathcal{D}}_{jl}^{\mathcal{X}})^c | X_{jl}). \end{aligned}$$

This last term is exponentially small with high probability by the previous considerations, which proves Lemma 14.

## APPENDIX F

### PROOF OF LEMMA 15

Below we will define events  $\iota_1(j, z^n, s^n)$  and  $\iota_2(j, s^n)$ , for  $j \in \mathcal{J}_n$ ,  $z^n \in \mathcal{Z}^n$  and  $s^n \in \mathcal{S}^n$ , and show that the  $\iota_0$  defined in Lemma 15 satisfies

$$\iota_0 \supset \bigcap_{j, z^n, s^n} \iota_1(j, z^n, s^n) \cap \bigcap_{j, s^n} \iota_2(j, s^n). \quad (58)$$

Then to show that  $\mathbb{P}[\iota_0] > 1 - 2^{-\tau_2 n}$ , it remains to prove that each of the events of the right-hand side of (58) has a probability sufficiently close to 1.

1) *Definition of  $\iota_1(j, z^n, s^n)$* : For some positive  $\alpha$  to be chosen later, let  $\varepsilon_n := 2^{-n\alpha}$ . Fix  $s^n \in \mathcal{S}^n$ , and denote its type by  $q \in \mathcal{P}_0^n(\mathcal{S})$ . For  $x^n \in \mathcal{A}^n$ , define

$$\mathcal{E}_1(x^n, s^n) := \{z^n \in \mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n : V_{s^n}^n(z^n|x^n) \leq \exp\{-n(H(\bar{Z}_q|\bar{X}) - f_2(3|\mathcal{S}|\delta))\}\},$$

where  $f_2$  is the function from Lemma 19, and set

$$\tilde{\Theta}_{s^n}(z^n) := \mathbb{E}[V_{s^n}^n(z^n|X_{11})\mathbb{1}_{\mathcal{E}_1(X_{11}, s^n)}(z^n)]. \quad (59)$$

Further define

$$\mathcal{E}_2(s^n) := \{z^n \in \mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n : \tilde{\Theta}_{s^n}(z^n) \geq \varepsilon_n |\mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n|^{-1}\}$$

and set

$$\Theta_{s^n}(z^n) := \tilde{\Theta}_{s^n}(z^n)\mathbb{1}_{\mathcal{E}_2(s^n)}(z^n).$$

Note that by definition,  $\Theta_{s^n}(z^n) > 0$  only if  $z^n \in \mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n$ .

With the sets just defined, we obtain a modification of  $V_{s^n}^n$  by defining

$$Q_{s^n, z^n}(x^n) := V_{s^n}^n(z^n|x^n)\mathbb{1}_{\mathcal{E}_1(x^n, s^n)}(z^n)\mathbb{1}_{\mathcal{E}_2(s^n)}(z^n).$$

Note that this is not an actual “channel” as in general  $\sum_{z^n} Q_{s^n, z^n}(x^n) < 1$ . Finally, we define

$$\iota_1(j, z^n, s^n) := \left\{ \frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, z^n}(X_{jl}) \in [(1 \pm \varepsilon_n)\Theta_{s^n}(z^n)] \right\},$$

where  $[(1 \pm \varepsilon_n)\Theta_{s^n}(z^n)]$  is short for  $[(1 - \varepsilon_n)\Theta_{s^n}(z^n), (1 + \varepsilon_n)\Theta_{s^n}(z^n)]$ .

2) *Definition of  $\iota_2(j, s^n)$* : Let  $q \in \mathcal{P}_0^n(\mathcal{S})$  be the type of  $s^n$  and let  $\bar{S}_q$  be an  $\mathcal{S}$ -valued random variable independent of the family  $\{\bar{X}, \bar{Y}_r, \bar{Z}_q : r \in \mathcal{R}, q \in \mathcal{P}(\mathcal{S})\}$  with  $P_{\bar{S}_q} = q$ . Then we define

$$\iota_2(j, s^n) := \left\{ |\{l \in \mathcal{L}_n : s^n \in T_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{jl})\}| \geq (1 - \varepsilon_n)(1 - 2^{-nc'\delta^2})L_n \right\}.$$

3) *Proof of Lemma 15*: The proof of Lemma 15 bases on three lemmas. The first one proves that (58) actually is true.

*Lemma 24*: Assume a realization  $\mathbf{x} := \{x_{jl} : j \in \mathcal{J}_n, l \in \mathcal{L}_n\}$  of  $\mathcal{X}$  has the following properties: For all  $j \in \mathcal{J}_n$  and  $z^n \in \mathcal{C}^n$  and  $q \in \mathcal{P}_0^n(\mathcal{S})$  and  $s^n \in \mathcal{S}^n$ ,

$$\frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, z^n}(x_{jl}) \in [(1 \pm \varepsilon_n)\Theta_{s^n}(z^n)], \quad (60)$$

$$\frac{|\{l \in \mathcal{L}_n : s^n \in T_{\bar{S}_q, 2\delta}^n(x_{jl})\}|}{L_n} \geq (1 - \varepsilon_n - 2^{-nc'\delta^2}), \quad (61)$$

Then

$$\max_{j \in \mathcal{J}_n} \max_{s^n \in \mathcal{S}^n} \|P_{Z_{s^n}|M^n}(\cdot|j) - \Theta_{s^n}(\cdot)\| \leq 4(\varepsilon_n + 2^{-nc'\delta^2}).$$

In particular, (58) is true with  $\tau_1 = \min\{\alpha, c'\delta^2\}/2$ .

This lemma is proved in Appendix G. The next two lemmas bound the probabilities of the complements of the  $\iota_1$  and  $\iota_2$  sets.

*Lemma 25:* For sufficiently small  $\delta > 0$  there exists a  $\tau_3 > 0$  such that for  $n$  large and every  $j \in \mathcal{J}_n, z^n \in \mathcal{C}^n$  and  $s^n \in \mathcal{S}^n$

$$\mathbb{P}[\iota_1(j, z^n, s^n)^c] \leq 2 \exp\left\{-\exp\{n\tau_3\}\right\}.$$

*Lemma 26:* For every  $j \in \mathcal{J}_n$  and  $s^n \in \mathcal{S}^n$ ,

$$\mathbb{P}[\iota_2(j, s^n)] \leq 2 \exp\left\{-\exp\left\{n\left(\max_{q \in \mathcal{P}(\mathcal{S})} I(\bar{X} \wedge \bar{Z}_q) + \tau_5\right)\right\}\right\}$$

for some  $\tau_5 > 0$ , provided that  $n$  is sufficiently large.

The proofs of Lemmas 25 and 26 can be found in Appendix G. They show that the probability of the complement of each of the events  $\iota_1(j, z^n, s^n)$  and  $\iota_2(j, s^n)$  is upper-bounded by a term which tends to zero doubly-exponentially as the blocklength increases. Then

$$\begin{aligned} \mathbb{P}[\iota_0] &= 1 - \mathbb{P}[\iota_0^c] \\ &\stackrel{(i)}{\geq} 1 - \mathbb{P}\left[\bigcup_{j, z^n, s^n} \iota_1(j, z^n, s^n)^c \cup \bigcup_{j, s^n} \iota_2(j, s^n)^c\right] \\ &\stackrel{(ii)}{\geq} 1 - 2J_n |\mathcal{C}|^n |\mathcal{S}|^n \exp\left\{-\exp\{n\tau_3\}\right\} - 2J_n |\mathcal{S}|^n \exp\left\{-\exp\left\{n\left(\max_{q \in \mathcal{P}(\mathcal{S})} I(\bar{X} \wedge \bar{Z}_q) + \tau_5\right)\right\}\right\} \\ &\stackrel{(iii)}{\geq} 1 - 2^{-n\tau_1}, \end{aligned}$$

where (i) is due to (58), (ii) to the union bound and (iii) because an appropriate  $\tau_1 > 0$  can be found due to the doubly exponential decrease of the probabilities in Lemmas 25 and 26. Altogether, this proves Lemma 15.

## APPENDIX G

### PROOFS OF LEMMAS 24-26

#### A. Proof of Lemma 25

Let  $j \in \mathcal{J}_n, z^n \in \mathcal{C}^n, s^n \in \mathcal{S}^n$ . We want to upper-bound the probability of the event that

$$\left\{\frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, z^n}(X_{jl}) \notin [(1 \pm \varepsilon_n)\Theta_{s^n}(z^n)]\right\}.$$

The form of this event already suggests that a Chernoff bound may be the right method for the proof. Indeed, we will apply the following lemma.

*Lemma 27:* Let  $b$  be a positive number. Let  $Z_1, \dots, Z_L$  be i.i.d. random variables with values in  $[0, b]$  and expectation  $\mathbb{E}Z_l = \nu$ , and let  $0 < \varepsilon < \frac{1}{2}$ . Then

$$\mathbb{P}\left\{\frac{1}{L} \sum_{l=1}^L Z_l \notin [(1 \pm \varepsilon)\nu]\right\} \leq 2 \exp\left(-L \cdot \frac{\varepsilon^2 \nu}{3b}\right).$$

*Proof:* The proof can be found in [14, Theorem 1.1] and in [3]. ■

The claim of Lemma 25 follows from an application of Lemma 27. Due to the definition of  $\mathcal{E}_1(x^n, s^n)$ , the random variables  $Q_{s^n, z^n}(X_{jl})$  are upper-bounded by  $\exp\{-n(H(\bar{Z}_q|\bar{X}) - f_2(3|\mathcal{S}|\delta))\}$  and have mean  $\Theta_{s^n}(z^n)$ . Lemma 19 implies that  $\Theta_{s^n}(z^n) \geq \varepsilon_n \exp\{-n(H(\bar{Z}_q) + f_1(4|\mathcal{A}||\mathcal{S}|\delta))\}$ . Inserting this into the right-hand side of

Lemma 27 and recalling the definition of  $\varepsilon_n$  gives the desired bound, with  $\tau_3 = \tau/5 - 3\alpha - f_1(4|\mathcal{A}||\mathcal{S}|\delta) - f_2(3|\mathcal{S}|\delta)$ . This is positive if  $\alpha$  and  $\delta$  are sufficiently small. This proves Lemma 25.

### B. Proof of Lemma 26

The proof also applies the Chernoff bound of Lemma 27. To do so, we need a lower bound on  $\mathbb{E}[\mathbb{1}_{\mathcal{T}_{\bar{S}_q, 2\delta}^n(X_{11})}] = \mathbb{P}[s^n \in \mathcal{T}_{\bar{S}_q, 2\delta}^n(X_{11})]$ .

*Lemma 28:* For sufficiently large  $n$  and every  $s^n$  of type  $q$ ,

$$\mathbb{P}[s^n \in \mathcal{T}_{\bar{S}_q, 2\delta}^n(X_{11})] \geq 1 - 2^{-nc'\delta^2}.$$

*Proof:* We first show

$$\mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n) \subset \{x^n \in \mathcal{T}_{\bar{X}, \delta}^n : s^n \in \mathcal{T}_{\bar{S}_q, 2\delta}^n(x^n)\}. \quad (62)$$

Let  $x^n \in \mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n)$ . Clearly  $\mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n) \subset \mathcal{T}_{\bar{X}, \delta}^n$ . Then

$$\begin{aligned} & \left| \frac{1}{n} N(s, a | s^n, x^n) - P_{\bar{S}_q|\bar{X}}(s|a) \frac{1}{n} N(a | x^n) \right| \\ &= \left| \frac{1}{n} N(s, a | s^n, x^n) - \frac{1}{n} N(s | s^n) \frac{1}{n} N(a | x^n) \right| \\ &\leq \left| \frac{1}{n} N(s, a | s^n, x^n) - P_{\bar{X}|\bar{S}_q}(a|s) \frac{1}{n} N(s | s^n) \right| \\ &\quad + \left| \frac{1}{n} N(s | s^n) \left[ P_{\bar{X}}(a) - \frac{1}{n} N(a | x^n) \right] \right| \\ &\leq \frac{\delta}{|\mathcal{S}|} + \delta \leq 2\delta. \end{aligned}$$

This proves (62). For  $n$  large, we can use this to continue with

$$\begin{aligned} \mathbb{P}[s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})] &\stackrel{(i)}{\geq} \mathbb{P}[\mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n)] = \sum_{x^n \in \mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n)} p'(x^n) \\ &\stackrel{(ii)}{\geq} \sum_{x^n \in \mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n)} P_{\bar{X}}^n(x^n) \\ &= P_{\bar{X}|\bar{S}_q}^n(\mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n) | s^n) \\ &\stackrel{(iii)}{\geq} 1 - 2^{-nc'\delta^2}, \end{aligned}$$

where we used (62) in (i),  $\mathcal{T}_{\bar{X}|\bar{S}_q, \delta/|\mathcal{S}|}^n(s^n) \subset \mathcal{T}_{\bar{X}, \delta}^n$  in (ii) and Lemma 20 in (iii).  $\blacksquare$

Moving to the proof of Lemma 26, let  $j \in \mathcal{J}_n$ . The i.i.d. random variables  $\mathbb{1}_{\mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{jl})}(s^n)$  ( $l \in \mathcal{L}_n$ ) are upper-bounded by 1. Their expectation  $\nu$  was lower-bounded in Lemma 28 by  $1 - 2^{-nc'\delta^2}$ . This implies that  $\iota_2(j, s^n)^c$  is contained in the event

$$\left\{ \frac{1}{L_n} |\{l \in \mathcal{L}_n : s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{jl})\}| \leq (1 - \varepsilon_n)\nu \right\}.$$

Lemma 27 thus implies that the probability of the above event is upper-bounded as claimed if  $n$  is large enough upon setting  $\tau_5 := \tau/4 - 3\alpha$  and letting  $\alpha$  be small enough.

### C. Proof of Lemma 24

The next two lemmas are needed for the proof. Recall the convention that we sometimes write  $V(c|a, s)$  instead of  $V_s(c|a)$ .

*Lemma 29:* Let  $x^n \in \mathcal{T}_{\bar{X}, \delta}^n$  and let  $s^n$  have type  $q \in \mathcal{P}_0^n(\mathcal{S})$ . Let the random variable  $\underline{Z}_q$  satisfy  $P_{\underline{Z}_q|\bar{X}\bar{S}_q}(\cdot|\cdot, \cdot) = V(\cdot|\cdot, \cdot)$ . If  $s^n \in \mathcal{T}_{\bar{S}_q, 2\delta}^n(x^n)$ , then  $\mathcal{T}_{\underline{Z}_q|\bar{X}\bar{S}_q, \delta}^n(x^n, s^n) \subset \mathcal{E}_1(x^n, s^n)$ .

*Proof:* For  $x^n \in \mathcal{T}_{\bar{X}, \delta}^n$ , we have  $\mathcal{T}_{\bar{Z}_q|\bar{X}, 3|\mathcal{S}|\delta}^n(x^n) \subset \mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n$ . Thus due to Lemma 19, it suffices to show that if  $s^n$  has type  $q$ , then  $\mathcal{T}_{\underline{Z}_q|\bar{X}\bar{S}_q, \delta}^n(x^n, s^n) \subset \mathcal{T}_{\bar{Z}_q|\bar{X}, 3|\mathcal{S}|\delta}^n(x^n)$ . For  $a \in \mathcal{A}$  and  $c \in \mathcal{C}$ , we calculate

$$\begin{aligned} & \left| \frac{1}{n} N(c, a|z^n, x^n) - \sum_{s \in \mathcal{S}} q(s) V(c|a, s) \frac{1}{n} N(a|x^n) \right| \\ & \leq \sum_{s \in \mathcal{S}} \left| \frac{1}{n} N(c, a, s|z^n, x^n, s^n) - q(s) V(c|a, s) \frac{1}{n} N(a|x^n) \right| \\ & \leq \sum_{s \in \mathcal{S}} \left| \frac{1}{n} N(c, a, s|z^n, x^n, s^n) - V(c|a, s) \frac{1}{n} N(a, s|x^n, s^n) \right| \\ & \quad + \sum_{s \in \mathcal{S}} V(c|a, s) \left| \frac{1}{n} N(a, s|x^n, s^n) - q(s) \frac{1}{n} N(a|x^n) \right| \\ & \leq |\mathcal{S}|(\delta + 2\delta) = 3|\mathcal{S}|\delta. \end{aligned}$$

■

*Corollary 3:* If  $n$  is sufficiently large, then every  $s^n \in \mathcal{S}^n$  satisfies

$$\Theta_{s^n}(\mathcal{C}^n) \geq 1 - 2 \cdot 2^{-nc'\delta^2} - \varepsilon_n$$

*Proof:* Let  $s^n$  have type  $q \in \mathcal{P}_0^n(\mathcal{S})$ . By the definition of  $\Theta_{s^n}$ , we have  $\Theta_{s^n}(\mathcal{C}^n) = \Theta_{s^n}(\mathcal{E}_2(s^n))$ . As the support of  $\tilde{\Theta}_{s^n}$  is contained in  $\mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n$ , we have  $\Theta_{s^n}(\mathcal{E}_2(s^n)) \geq \tilde{\Theta}_{s^n}(\mathcal{T}_{\bar{Z}_q, 4|\mathcal{A}||\mathcal{S}|\delta}^n) - \varepsilon_n = \tilde{\Theta}_{s^n}(\mathcal{C}^n) - \varepsilon_n$ . By definition,

$$\begin{aligned} \tilde{\Theta}_{s^n}(\mathcal{C}^n) &= \mathbb{E}[V_{s^n}^n(\mathcal{E}_1(X_{11}, s^n)|X_{11})] \\ &\geq \mathbb{E}[V_{s^n}^n(\mathcal{E}_1(X_{11}, s^n)|X_{11})|s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})] \mathbb{P}[s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})]. \end{aligned}$$

For sufficiently large  $n$

$$\begin{aligned} & \mathbb{E}[V_{s^n}^n(\mathcal{E}_1(X_{11}, s^n)|X_{11})|s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})] \\ & \stackrel{(i)}{\geq} \mathbb{E}[V^n(\mathcal{T}_{\underline{Z}_q|\bar{X}\bar{S}_q, \delta}^n(X_{11}, s^n)|X_{11}, s^n)|s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})] \\ & \stackrel{(ii)}{\geq} 1 - 2^{-nc'\delta^2}, \end{aligned}$$

where we used Lemma 29 in (i) and Lemma 20 in (ii). Lemma 28 provides a lower bound on  $\mathbb{P}[s^n \in \mathcal{T}_{\bar{S}_q|\bar{X}, 2\delta}^n(X_{11})]$ , so altogether,

$$\Theta_{s^n}(\mathcal{C}^n) \geq \tilde{\Theta}_{s^n}(\mathcal{C}^n) - \varepsilon_n \geq (1 - 2^{-nc'\delta^2})^2 - \varepsilon_n \geq 1 - 2 \cdot 2^{-nc'\delta^2} - \varepsilon_n. \quad (63)$$

■



Let  $\mathbf{x} = \{x_{jl} : j \in \mathcal{J}_n, l \in \mathcal{L}_n\}$  be a realization of  $\mathcal{X}$  satisfying (60) and (61). Let  $\mathcal{K}_n$  be the corresponding code and  $\mathcal{F}(\mathcal{K}_n, \overline{\mathcal{W}}, \mathfrak{V}) = \{M^n, X^n, Y_r^n, Z_{s^n}^n, \hat{M}_r : r \in \mathcal{R}, s^n \in \mathcal{S}^n\}$  the canonical family of random variables associated with  $\mathcal{K}_n$ . For any  $s^n$  with type  $q \in \mathcal{P}_0(\mathcal{S})$ , we decompose the total variation distance as follows:

$$\begin{aligned} & \|P_{Z_{s^n}^n | M^n}(\cdot | j) - \Theta_{s^n}(\cdot)\| \\ & \leq \left\| \frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, \cdot}(x_{jl}) - \Theta_{s^n}(\cdot) \right\| \end{aligned} \quad (64)$$

$$+ \left\| \frac{1}{L_n} \sum_{l=1}^{L_n} V_{s^n}^n(\cdot | x_{jl}) \mathbb{1}_{\mathcal{E}_1(x_{jl}, s^n)}(\cdot) (\mathbb{1}_{\mathcal{C}^n}(\cdot) - \mathbb{1}_{\mathcal{E}_2(s^n)}(\cdot)) \right\| \quad (65)$$

$$+ \left\| \frac{1}{L_n} \sum_{l=1}^{L_n} V_{s^n}^n(\cdot | x_{jl}) (\mathbb{1}_{\mathcal{C}^n}(\cdot) - \mathbb{1}_{\mathcal{E}_1(x_{jl}, s^n)}(\cdot)) \right\|. \quad (66)$$

The term in (64) is upper-bounded by  $\varepsilon_n$ , because due to (60)

$$\begin{aligned} & \left\| \frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, \cdot}(x_{jl}) - \Theta_{s^n}(\cdot) \right\| \\ & = \sum_{z^n} \left| \frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, z^n}(x_{jl}) - \Theta_{s^n}(z^n) \right| \\ & \leq \varepsilon_n \sum_{z^n} \Theta_{s^n}(z^n) \\ & \leq \varepsilon_n. \end{aligned}$$

Next, applying (60) in (i), we upper-bound (65) as

$$\begin{aligned} & \frac{1}{L_n} \sum_{l=1}^{L_n} \sum_{z^n} V_{s^n}(z^n | x_{jl}) \mathbb{1}_{\mathcal{E}_1(x_{jl}, s^n)}(z^n) \\ & - \frac{1}{L_n} \sum_{l=1}^{L_n} \sum_{z^n} V_{s^n}(z^n | x_{jl}) \mathbb{1}_{\mathcal{E}_1(x_{jl}, s^n)}(z^n) \mathbb{1}_{\mathcal{E}_2(s^n)}(z^n) \\ & \leq 1 - \sum_{z^n} \frac{1}{L_n} \sum_{l=1}^{L_n} Q_{s^n, z^n}(x_{jl}) \\ & \stackrel{(i)}{\leq} 1 - (1 - \varepsilon_n) \Theta_{s^n}(\mathcal{C}^n). \end{aligned}$$

Upon application of Corollary 3, we obtain that (65) can be upper-bounded by

$$1 - (1 - \varepsilon_n)(1 - 2 \cdot 2^{-nc'\delta^2} - \varepsilon_n) \leq 2(2^{-nc'\delta} + \varepsilon_n).$$

It remains to upper-bound (66). Recall the definition of  $\underline{Z}_q$ . We have

$$\begin{aligned}
& \left\| \frac{1}{L_n} \sum_{l=1}^{L_n} V_{s^n}^n(\cdot | x_{jl}) (\mathbb{1}_{\mathcal{C}^n}(\cdot) - \mathbb{1}_{\mathcal{E}_1(x_{jl}, s^n)}(\cdot)) \right\| \\
&= \frac{1}{L_n} \sum_{l=1}^{L_n} V_{s^n}^n(\mathcal{E}_1(x_{jl}, s^n)^c | x_{jl}) \\
&= \frac{1}{L_n} \sum_{\substack{l \in \mathcal{L}_n: \\ \mathcal{T}_{\underline{Z}_q | \bar{X} \bar{S}_q, \delta}^n(x_{jl}, s^n) \subset \mathcal{E}_1(x_{jl}, s^n)}} V_{s^n}^n(\mathcal{E}_1(x_{jl}, s^n)^c | x_{jl}) \\
&+ \frac{1}{L_n} \sum_{\substack{l \in \mathcal{L}_n: \\ \mathcal{T}_{\underline{Z}_q | \bar{X} \bar{S}_q, \delta}^n(x_{jl}, s^n) \not\subset \mathcal{E}_1(x_{jl}, s^n)}} V_{s^n}^n(\mathcal{E}_1(x_{jl}, s^n)^c | x_{jl}).
\end{aligned} \tag{67}$$

If  $\mathcal{T}_{\underline{Z}_q | \bar{X} \bar{S}_q, \delta}^n(x_{jl}, s^n) \subset \mathcal{E}_1(x_{jl}, s^n)$ , then by Lemma 20, we have

$$V_{s^n}^n(\mathcal{E}_1(x_{jl}, s^n)^c | x_{jl}) \leq V^n(\mathcal{T}_{\underline{Z}_q | \bar{X} \bar{S}_q, \delta}^n(x_{jl}, s^n)^c | x_{jl}, s^n) \leq 2^{-nc'\delta^2}.$$

By Lemma 29 and (61), the proportion of those  $j$  for which  $\mathcal{T}_{\underline{Z}_q | \bar{X} \bar{S}_q, \delta}^n(x_{jl}, s^n) \not\subset \mathcal{E}_1(x_{jl}, s^n)$  holds is upper-bounded by  $\varepsilon_n + 2^{-nc'\delta^2}$ . We can thus bound (67) by

$$2^{-nc'\delta^2} + \varepsilon_n + 2^{-nc'\delta} = \varepsilon_n + 2 \cdot 2^{-nc'\delta^2}.$$

Collecting the bounds on (64), (65) and (66) completes the proof of Lemma 24.

## REFERENCES

- [1] R. Ahlswede. Elimination of correlation in random codes for arbitrarily varying channels. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 44:159–175, 1978.
- [2] R. Ahlswede. Arbitrarily varying channels with states sequence known to the sender. *IEEE Trans. Inf. Theory*, IT-32(5):621–629, 1986.
- [3] R. Ahlswede and A. Winter. Strong converse for identification via quantum channels. *IEEE Trans. Inf. Theory*, 48(3):569–579, 2002.
- [4] I. Bjelaković, H. Boche, and J. Sommerfeld. Capacity results for arbitrarily varying wiretap channels. In Harout Aydinian, Ferdinando Cicalese, and Christian Deppe, editors, *Information Theory, Combinatorics, and Search Theory*, volume 7777 of *Lecture Notes in Computer Science*, pages 123–144. Springer Berlin Heidelberg, 2013.
- [5] I. Bjelaković, H. Boche, and J. Sommerfeld. Secrecy results for compound wiretap channels. *Problems of Information Transmission*, 49(1):73–98, 2013.
- [6] D. Blackwell, L. Breiman, and A. J. Thomasian. The capacity of a class of channels. *Ann. Math. Statist.*, 30(4):1229–1241, 1959.
- [7] D. Blackwell, L. Breiman, and A. J. Thomasian. The capacities of certain channel classes under random coding. *Ann. Math. Statist.*, 31(3):558–567, 1960.
- [8] M. R. Bloch and J. N. Laneman. Strong Secrecy From Channel Resolvability. *IEEE Trans. Inf. Theory*, 59(12):8077–8098, 2013.
- [9] H. Boche, R. F. Schaefer, and H. V. Poor. On the continuity of the secrecy capacity of compound and arbitrarily varying wiretap channels. Available at <http://arxiv.org/abs/1409.4752>, October 2014.
- [10] H. Boche and R. F. Schaefer. Arbitrarily varying wiretap channels with finite coordination resources. In *Communications Workshops (ICC), 2014 IEEE International Conference on*, pages 746–751, June 2014.
- [11] I. Csiszár and J. Körner. *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Cambridge University Press, Cambridge, second edition, 2011.
- [12] I. Csiszár and P. Narayan. The capacity of the arbitrarily varying channel revisited: positivity, constraints. *IEEE Trans. Inf. Theory*, 34(2):181–193, mar 1988.
- [13] I. Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Trans. Inf. Theory*, 51(1):44–55, 2005.

- [14] D. D. Dubhashi and A. Panconesi. *Concentration of Measure for the Analysis of Randomized Algorithms*. Cambridge University Press, 2012.
- [15] T. Ericson. Exponential error bounds for random codes in the arbitrarily varying channel. *IEEE Trans. Inf. Theory*, 31(1):42–48, 1985.
- [16] M. Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Mathematische Zeitschrift*, 17(1):228–249, 1923.
- [17] T. S. Han. *Information-Spectrum Methods in Information Theory*. Springer-Verlag Berlin Heidelberg 2003.
- [18] X. He, A. Khisti and A. Yener. MIMO Multiple Access Channel With an Arbitrarily Varying Eavesdropper: Secrecy Degrees of Freedom. *IEEE Trans. Inf. Theory*, 59(8):4733–4745, 2013.
- [19] X. He and A. Yener. MIMO Wiretap Channels With Unknown and Varying Eavesdropper Channel States. *IEEE Trans. Inf. Theory*, 60(11):6844–6869, 2014.
- [20] E. MolavianJazi, M. Bloch, and J.N. Laneman. Arbitrary jamming can preclude secure communication. In *Communication, Control, and Computing, 2009. Allerton 2009. 47th Annual Allerton Conference on*, pages 1069–1075, Sept 2009.
- [21] J. Nötzel, M. Wiese, and H. Boche. The Arbitrarily Varying Wiretap Channel – Secret Randomness, Stability and Super-Activation. Available at <http://arxiv.org/abs/1501.07439>, January 2015.
- [22] C. E. Shannon. A mathematical theory of communication. *Bell Syst. Tech. J.*, 27:379–423, 623–656, 1948.
- [23] M. Wiese and H. Boche. Strong secrecy for multiple access channels. In Harout Aydinian, Ferdinando Cicalese, and Christian Deppe, editors, *Information Theory, Combinatorics, and Search Theory*, volume 7777 of *Lecture Notes in Computer Science*, pages 71–122. Springer Berlin Heidelberg, 2013.